Imprecise Reliability
(tutorial)

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Reliability *is the measurable capability of a system to perform its intended function in the required time under specified conditions.*

Reliability index *is a quantitative measure of reliability properties.*

$$\Pr\{X > Y\},\ H(t) = \Pr\{X > t\},\ \mathbb{E}X, \ldots$$

Two main tasks in reliability analysis

Two different aspects (problems, parts) of reliability analysis can be selected:

1. **Statistical inference of the component (system) reliability measures**
   - *Standard methods of statistical inference, regression analysis, etc. for computing reliability measures from statistical data and expert judgments*
2. **System reliability analysis**
   - *Probabilities (expectations) of a function of random times to failure.*
3. Some specific problems of reliability and risk analyses

System reliability analysis

If there is a vector of n random variables $\widetilde{X} = (X_1, ..., X_n)$:

- unit times to failure for a system of n units,
- load or stress factors for a structural systems,
- switch times, times to repair, etc.

and a system reliability is defined as a function $Y = g(\widetilde{X})$:

- system time to failure,
- stress minus strength ($g(\widetilde{X}) = X_1 - X_2$), etc.

then our goal is to compute reliability indices

$$\Pr\{g(\widetilde{X}) > t\},\ \mathbb{E}g(\widetilde{X}), \ldots$$

under two assumptions.

Two main assumptions

1. all probabilities of events or probability distributions of r.v. $X_1, ..., X_n$ are known or perfectly determinable;
2. the system units or r.v. $X_1, ..., X_n$ are statistically independent or their dependence is precisely known.

The assumptions are usually not fulfilled. As a result, the reliability may be too unreliable and risky.

Alternative approaches

1. Interval reliability (interval probabilities in the framework of standard interval calculation) /*not interesting*/.
2. Fuzzy (possibilistic) reliability as an extension of interval models (Cai et al. 1996, de Cooman 1996, Utkin-Gurov 1996, etc.). The models have many "open" questions:
    - How to deal with the large imprecision by analyzing large systems?
    - How to interpret the possibilistic reliability measures?
    - How to take into account conditions of independence?
    - How how how ...?
3. *Reliability in the framework of random sets and evidence theories (Hall and Lawry 2001,Tonon, Bernardini, Elishakoff 1999, Oberguggenberger, Fetz, Pittschmann 2000, etc.)*

Elements of imprecise reliability in the classical approach

1. An attempt to consider sets of distributions:
   - *ageing* aspects of lifetime distributions, in particular, IFRA (increasing failure rate average) and DFRA (decreasing failure rate average) distributions (Barlow and Proschan 1975);
   - various nonparametric or semi-parametric classes of probability distributions (Barzilovich and Kashtanov 1971).
2. An attempt to use some models of joint probability distributions for taking into account the lack of independence:
   - Frechet bounds for series systems ($Y = \min(X_1, ..., X_n)$) (Barlow and Proschan 1975).
3. An attempt to use bounds for system reliability (Lindqvist and Langseth 1998).

First steps in imprecise reliability

The first works with explicit indication on the **imprecise** approach in reliability:

- Imprecise probabilities in a generalized Bayesian statistical framework (Coolen 1994, 1996, 1997).
- Imprecise probabilities in system reliability analysis (Utkin and Gurov 1999, 2001, Kozine 1999, Kozine and Filimonov 2000, Kozine and Utkin 2000)

System reliability as the natural extension (1)

Information about units or r.v. $X_1, ..., X_n$:

- unit 1: m_1 interval-valued estimates: $[\underline{\mathbf{E}}f_{1j}, \overline{\mathbf{E}}f_{1j}]$, $j = 1, .., m_1$;
- unit 2: m_2 interval-valued estimates: $[\underline{\mathbf{E}}f_{2j}, \overline{\mathbf{E}}f_{2j}]$, $j = 1, .., m_2$;

  . . .
- unit n: m_n interval-valued estimates: $[\underline{\mathbf{E}}f_{nj}, \overline{\mathbf{E}}f_{nj}]$, $j = 1, .., m_n$.

Target reliability measures:

$$\underline{\mathbf{E}}h(Y), \quad \overline{\mathbf{E}}h(Y), \text{ where } Y = g(X_1, ..., X_n).$$

System reliability as the natural extension (2)

Optimization problems:

$$\begin{aligned}\underline{\mathbf{E}}h(g) &= \min_{\mathcal{P}} \int_{\Omega^n} h(g(\mathbf{X}))\rho(\mathbf{X})\mathrm{d}\mathbf{X},\\ \overline{\mathbf{E}}h(g) &= \max_{\mathcal{P}} \int_{\Omega^n} h(g(\mathbf{X}))\rho(\mathbf{X})\mathrm{d}\mathbf{X},\end{aligned}$$

subject to

$$\begin{aligned}\rho(\mathbf{X}) &\geq 0,\ \int_{\Omega^n} \rho(\mathbf{X})\mathrm{d}\mathbf{X} = 1,\ \ \mathbf{X} = (x_1, ..., x_n),\\ \underline{\mathbf{E}}f_{ij} &\leq \int_{\Omega^n} f_{ij}(x_i)\rho(\mathbf{X})\mathrm{d}\mathbf{X} \leq \overline{\mathbf{E}}f_{ij},\ \ i \leq n,\ j \leq m_i.\end{aligned}$$

Independence $\rho(\mathbf{X}) = \rho_1(x_1) \times \cdots \times \rho_n(x_n)$

System reliability as the natural extension (3)

Dual optimization problems (Walley's and Kuznetsov's forms):

$$\underline{\mathbf{E}}h(g) = \max\left\{ c + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}\underline{\mathbf{E}}f_{ij} - d_{ij}\overline{\mathbf{E}}f_{ij}\right)\right\},$$

subject to $c_{ij}, d_{ij} \in \mathbf{R}_+$, $i = 1, ..., n$, $j = 1, ..., m_i$, $c \in \mathbf{R}$, and $\forall \mathbf{X} \in \Omega^n$,

$$c + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij} - d_{ij}\right) f_{ij} \leq h(g(\mathbf{X})).$$

There are other forms of the optimization problems, for instance, using Dirac functions as an analogue of extreme points (see Utkin and Kozine 2001).

System reliability as the natural extension (an example)

A two-unit series system:

Unit 1: the probability of failure before 10 hours is 0.01:

$$\underline{\mathbf{E}} I_{[0,10]}(X_1) = \overline{\mathbf{E}} I_{[0,10]}(X_1) = 0.01;$$

Unit 2: the mean time to failure is between 50 and 60 hours:

$$\underline{\mathbf{E}} X_2 = 50, \ \overline{\mathbf{E}} X_2 = 60.$$

What is the probability of system failure after time 100 **hours?**

$$\underline{\mathbf{E}} h(g) = ? \ h(g) = I_{[100,\infty)}(\min(X_1, X_2)).$$

System reliability as the natural extension (an example)

Optimization problems:

$$\underline{\mathbf{E}}h(g)\ (\overline{\mathbf{E}}h(g)) = \min_{\mathcal{P}} \left(\max_{\mathcal{P}} \right) \int_{\mathbf{R}_+^2} I_{[100,\infty)}(\min(x_1, x_2))\rho(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

subject to

$$0.01 \leq \int_{\mathbf{R}_+^2} I_{[0,10]}(x_1)\rho(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \leq 0.01,$$

$$50 \leq \int_{\mathbf{R}_+^2} x_2\rho(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \leq 60.$$

Solutions:

$\underline{\mathbf{E}}h(g) = 0$ and $\overline{\mathbf{E}}h(g) = 0.59$ (independent units),
$\underline{\mathbf{E}}h(g) = 0$ and $\overline{\mathbf{E}}h(g) = 0.99$ (no information about independence)

Why does not everyone use imprecise reliability?

The problems of classical reliability are obvious and were obvious always, but the above attempts to extend the classical reliability theory meets some trade-offs or conflict between:

- *simplicity and complexity,*
- *engineers (practitioners) and academic researchers,*
- *optimists and realists (pessimists),*
- *precision and imprecision,*
- *clear and unclear interpretations.*

What should be done in the light of the above questions?

1. To provide clear procedures for obtaining the unit reliability measures based on statistical data and expert judgments.

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2. To provide simple algorithms for special problems or systems.
3. To provide the clear interpretation of "imprecise" reliability measures.

What should be done in the light of the above questions?

1. To provide clear procedures for obtaining the unit reliability measures based on statistical data and expert judgments.
2. To provide simple algorithms for special problems or systems.
3. To provide the clear interpretation of "imprecise" reliability measures.
4. To popularize "imprecise" ideas among engineers.

Typical initial data or reliability measures in standard reliability analysis

1. Points of lifetime distributions or CDFs of time to failure.
2. Lifetime distributions.
3. Mean time to failure (to repair): the expectation of the random time to failure (to repair).
4. Residual measures: conditional expectations.

Corresponding imprecise initial data or reliability measures

1. Interval-valued points of lifetime distributions or CDFs of time to failure $\Pr\{X < t\} \in [\underline{p} \subset, \overline{p}]$.
2. P-boxes $\underline{F}(t) \leq F(t) \leq \overline{F}(t)$, $\forall t \geq 0$.
3. Interval-valued mean time to failure (to repair) $\underline{\mathbf{E}}X$ and $\overline{\mathbf{E}}X$.
4. Imprecise conditional expectations.
5. Second-order models $\Pr\left\{\underline{\mathbf{E}}f(X) \leq \mathbf{E}f(X) \leq \overline{\mathbf{E}}f(X)\right\} \in [\underline{\gamma}, \overline{\gamma}]$.
6. Possibilistic information: $\Pr\{\underline{t}_i \leq X \leq \overline{t}_i\} = p_i$, $[\underline{t}_i, \overline{t}_i] \subseteq [\underline{t}_{i+1}, \overline{t}_{i+1}]$, $p_i \leq p_{i+1}$ (nested intervals).

P-boxes and confidence intervals on the mean and variance

95% confidence intervals on the mean and variance ($\alpha = 0.05$):

$$[\underline{\mu}, \overline{\mu}] = [\hat{\mu} - t_{\alpha/2,N-1}\hat{\sigma}/\sqrt{N},\ \hat{\mu} + t_{\alpha/2,N-1}\hat{\sigma}/\sqrt{N}],$$

$$[\underline{\sigma}^2, \overline{\sigma}^2] = \left[\frac{(N-1)\hat{\sigma}^2}{\chi^2{}_{\alpha/2,N-1}},\ \frac{(N-1)\hat{\sigma}^2}{\chi^2{}_{1-\alpha/2,N-1}}\right],$$

$$\begin{aligned} \underline{F}(x) &= \min\left\{\Phi\left((x-\overline{\mu})/\overline{S}\right), \Phi\left((x-\overline{\mu})/\underline{S}\right)\right\}, \\ \overline{F}(x) &= \max\left\{\Phi\left((x-\underline{\mu})/\overline{S}\right), \Phi\left((x-\underline{\mu})/\underline{S}\right)\right\}. \end{aligned}$$

P-boxes and one-tailed version of Chebyshev's inequality

One-tailed version ($t \geq 0$) of Chebyshev's inequality:

$$\underline{F}(t) = \begin{cases} 0, & t < \overline{\mu} \\ 1 - \dfrac{\overline{\mu}}{t}, & \overline{\mu} \leq t \leq \overline{\mu} + \dfrac{\overline{\sigma}^2}{\overline{\mu}} \\ \dfrac{(\overline{\mu} - t)^2}{(\overline{\mu} - t)^2 + \overline{\sigma}^2}, & t > \overline{\mu} + \dfrac{\overline{\sigma}^2}{\overline{\mu}} \end{cases},$$

$$\overline{F}(t) = \begin{cases} \dfrac{\overline{\sigma}^2}{(\underline{\mu} - t)^2 + \overline{\sigma}^2}, & t < \underline{\mu} \\ 1, & t \geq \underline{\mu} \end{cases}.$$

P-boxes and imprecise Bayesian inference models

1. Imprecise Dirichlet model (Walley 1996);
2. Imprecise models for inference in exponential families (Quaeghebeur and de Cooman 2005).
3. Nonparametric predictive inference (Coolen)

H(r,s)-classes

- Survivor functions: $H(t) = \Pr(X \geq t) = \exp(-\Lambda(t))$,
- where $\Lambda(t) = \int_0^t \lambda(x)\mathrm{d}x$, $\lambda(t)$ is the failure rate.
- IFRA (increasing failure rate average): $\lambda(t)$ increases;
- DFRA (decreasing failure rate average): $\lambda(t)$ decreases.
- **IFRA and DFRA distribution classes are too large and imprecise!**
- **But they are natural and clear for reliability community!**

H(r,s)-classes

Definition

Let r and s be the numbers such that $0 \leq r < s \leq +\infty$. A probability distribution belongs to $\mathcal{H}(r,s)$ if $\Lambda(t)/t^r$ increases and $\Lambda(t)/t^s$ decreases as t increases.

1. $\mathcal{H}(1,+\infty)$ is the class of all IFRA distributions;
2. $\mathcal{H}(0,1)$ is the class of all DFRA distributions;
3. If $r_1 \leq r_2 \leq s_2 \leq r_1$, then $\mathcal{H}(r_1,s_1) \subset \mathcal{H}(r_2,s_2)$.
4. $r = s$ iff $H(t)$ is the Weibull distribution with the scale parameter λ and shape parameter r (s).
5. The Gamma distribution $\rho(t) = \lambda^k t^{k-1} e^{-\lambda t}/\Gamma(k)$ belongs to $\mathcal{H}(1,k)$ by $k \geq 1$ and to $\mathcal{H}(k,1)$ by $k < 1$.

P-boxes from H(r,s) with known bounds for the mean time to failure

Theorem

Suppose that the mean time to failure is in $[d_1, d_2]$*. The lower lifetime distribution is*

$$\underline{H}(t) = \underline{P}_{d_1,d_2}(0,t) = \min_{d_1 \leq d \leq d_2} e^{-\beta},$$

where β *is a solution of*

$$\frac{1}{s\beta^{1/s}}\Gamma(\beta, 1/s) - \frac{1}{r\beta^{1/r}}\Gamma(\beta, 1/r) + \frac{1}{r\beta^{1/r}}\Gamma(1/r) = \frac{d_1}{t}.$$

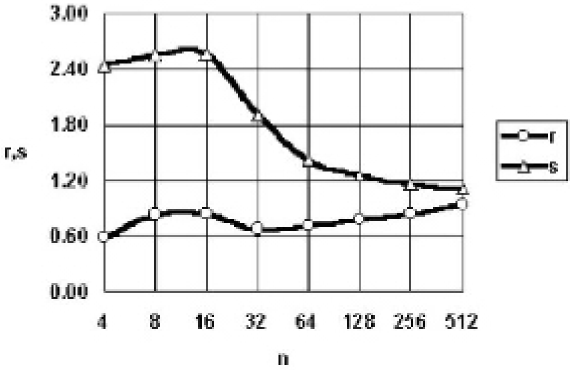
The main problem is how to find parameters r and s from statistical data

Given times to failure $\mathbf{T} = (t_1, ..., t_n)$.
a is the sample average from $\mathbf{T}$;
$\underline{\rho}_a(t_i), \overline{\rho}_a(t_i)$ are lower and upper density functions (from $\underline{H}(t)$ and $\overline{H}(t)$) given a;
A heuristic procedure: the "randomly averaged" likelihood function

$$L(\mathbf{T}|r,s) = \prod_{i:z_i=1} \overline{\rho}_a(t_i) \prod_{i:z_i=0} \underline{\rho}_a(t_i) \longrightarrow \max_{r,s}.$$

$$z_i \in \{0,1\},\ \Pr\{z=0\} = \Pr\{z=1\} = 0.5.$$

The main problem is how to find parameters r and s from statistical data



Residual reliability measures (failure rates, residual mean times to failure, etc.)

X_t is the residual lifetime, $X_t = X - t$ under condition $X \geq t$. The residual lifetime distribution $H_t(z)$ is

$$H_t(z) = \Pr\{X_t \geq z | X \geq t\} = \frac{\Pr\{X \geq z + t\}}{\Pr\{X \geq t\}} = \frac{H(z+t)}{H(t)}.$$

The residual mean time to failure $\mathbf{E}X_t$ is

$$\mathbf{E}X_t = \int_t^{\infty} H_t(z)\mathrm{d}z = \left(\int_t^{\infty}(x-t)\rho(x)\mathrm{d}x\right) / \left(\int_t^{\infty}\rho(x)\mathrm{d}x\right).$$

In the imprecise case, they can be computed by means of: Walley's generalized Bayes rule, Kuznetsov's rule with cut-previsions and linear-fractional programming.

Typical systems

1. A system is called **series** if it fails when at least one unit fails, $Y = \min_{i=1,\dots,n} X_i$.
2. A system is called **parallel** if it fails when all units fail, $Y = \max_{i=1,\dots,n} X_i$.
3. A m-**out-of**-n system fails if $m+1$ its components fails $(0 \leq m \leq n-1)$.
4. A **cold-standby** system $\left(Y = \sum_{i=1}^{n} X_i\right)$: If each component may have three states: operating, idle, and failed. At any time only one operative component is required and the other operatives are redundant. A failed component is replaced by a redundant component. A system failure occurs when no operative component is available.

Reliability of a series system (interval-valued points of lifetime distributions)

Independent units:

$$\begin{aligned} \underline{F}(t) &= \underline{\mathbf{E}} I_{[0,t]}(Y) = 1 - \prod_{i=1}^{n} \left(1 - \underline{F}_i(t_{iw_i})\right), \\ \overline{F}(t) &= \overline{\mathbf{E}} I_{[0,t]}(Y) = 1 - \prod_{i=1}^{n} \left(1 - \overline{F}_i(t_{iv_i})\right). \end{aligned}$$

A lack of knowledge about independence:

$$\begin{aligned} \underline{F}(t) &= \underline{\mathbf{E}} I_{[0,t]}(Y) = \max_{i=1,\dots,n} \underline{F}_i(t_{iw_i}), \\ \overline{F}(t) &= \overline{\mathbf{E}} I_{[0,t]}(Y) = \min\left(1, \sum_{i=1}^{n} \overline{F}_i(t_{iv_i})\right). \end{aligned}$$

Here $v_i = \min\{j : t_{ij} \geq t\}$ and $w_i = \max\{j : t_{ij} \leq t\}$.

Reliability of a parallel system (interval-valued points of lifetime distributions)

Independent units:

$$\underline{F}(t) = \underline{\mathbf{E}} I_{[0,t]}(Y) = \prod_{i=1}^{n} \underline{F}_i(t_{iw_i}),$$
$$\overline{F}(t) = \overline{\mathbf{E}} I_{[0,t]}(Y) = \prod_{i=1}^{n} \overline{F}_i(t_{iv_i}).$$

A lack of knowledge about independence:

$$\underline{F}(t) = \underline{\mathbf{E}} I_{[0,t]}(Y) = \max\left\{0, \sum_{i=1}^{n} \underline{F}_i(t_{iw_i}) - (n-1)\right\},$$
$$\overline{F}(t) = \overline{\mathbf{E}} I_{[0,t]}(Y) = \min_{i=1,\dots,n} \overline{F}_i(t_{iv_i}).$$

Here $v_i = \min\{j : t_{ij} \geq t\}$ and $w_i = \max\{j : t_{ij} \leq t\}$.

Reliability of a m-out-of-n system with identical units (interval-valued points of lifetime distributions)

Independent units:

$$\underline{F}(t) = \underline{\mathbf{E}} I_{[0,t]}(Y) = \sum_{i=m+1}^{n} \binom{n}{i} \underline{F}^{i}(t_w)(1-\underline{F}(t_w))^{n-i},$$
$$\overline{F}(t) = \overline{\mathbf{E}} I_{[0,t]}(Y) = \sum_{i=m+1}^{n} \binom{n}{i} \overline{F}^{i}(t_v)(1-\overline{F}(t_v))^{n-i}.$$

A lack of knowledge about independence:

$$\underline{F}(t) = \underline{\mathbf{E}} I_{[0,t]}(Y) = \max\{0, (m+1)\underline{F}(t_w) - m\},$$
$$\overline{F}(t) = \overline{\mathbf{E}} I_{[0,t]}(Y) = \min\left(1, (n-m)\overline{F}(t_v)\right).$$

Reliability of a cold-standby system (interval-valued points of lifetime distributions)

A lack of knowledge about independence:

$$\underline{F}(t) = \max_{V} \max \left\{ \sum_{i=1}^{n} \underline{F}_i(t_{iv_i}) - (n-1), 0 \right\},$$

$$\overline{F}(t) = \min \left\{ \min_{S} \min_{i=1,\dots,n} \overline{F}_i(t_{is_i}), \min_{W} \min \left(1, \sum_{i=1}^{n} \overline{F}_i(t_{i(w_i-1)}) \right) \right\}.$$

$$V = \left\{ (v_1, .., v_n) : \sum_{i=1}^{n} t_{iv_i} \leq t, \ v_i \in \{1, .., m_i + 1\} \right\},$$

$$W = \left\{ (w_1, .., w_n) : \sum_{i=1}^{n} t_{i(w_i-1)} \geq t, \ w_i \in \{1, .., m_i + 1\} \right\},$$

$$S = \left\{ (s_1, .., s_n) : t_{is_i} \geq t, \ s_i \in \{1, .., m_i + 1\} \right\},$$

Reliability of parallel and series systems (mean times to failure)

Given interval-valued mean time to failure (expectation of time to failure) of each unit: $\underline{\mathbf{E}}(X_i)$ and $\overline{\mathbf{E}}(X_i)$, $i = 1, ..., n$.
Parallel system:

$$\overline{\mathbf{E}}(Y) = \sum_{i=1}^{n} \overline{\mathbf{E}}(X_i), \ \underline{\mathbf{E}}(Y) = \max_{i=1,...,n} \underline{\mathbf{E}}(X_i).$$

Series system:

$$\overline{\mathbf{E}}(Y) = \min_{i=1,...,n} \overline{\mathbf{E}}(X_i), \ \underline{\mathbf{E}}(Y) = 0.$$

The condition of independence does not influence on the lower and upper system mean times to failure.

Reliability of a m-out-of-n system (mean times to failure)

Given interval-valued mean time to failure (expectation of time to failure) of each unit: $\underline{\mathbf{E}}(X_i)$ and $\overline{\mathbf{E}}(X_i)$, $i = 1, ..., n$.
Independent units and the lack of knowledge about independence:

$$\overline{\mathbf{E}}(Y) = \min_{\substack{i_1 < i_2 < ... < i_k \\ m+1 \leq k \leq n}} \frac{\sum_{j=1}^{k} \overline{\mathbf{E}}(X_{i_j})}{k - m},$$

$$\underline{\mathbf{E}}(Y) = \begin{cases} 0, & \text{if } m + 1 < n, \\ \max_{1 \leq i \leq n} \underline{\mathbf{E}}(X_i), & \text{if } m + 1 = n, \end{cases}.$$

The condition of independence does not influence on the lower and upper system mean times to failure.

Possibilistic time to failure

Given interval-valued probabilities $[\underline{p}_{ij}, \overline{p}_{ij}]$ of nested intervals $[\underline{t}_{ij}, \overline{t}_{ij}]$:

$$\underline{p}_{ij} \leq \Pr\{\underline{t}_{ij} \leq X_i \leq \overline{t}_{ij}\} \leq \overline{p}_{ij},\ i \leq n,\ j \leq m_i,$$

$$[\underline{t}_{i1}, \overline{t}_{i1}] \subset [\underline{t}_{i2}, \overline{t}_{i2}] \subset ... \subset [\underline{t}_{im_i}, \overline{t}_{im_i}].$$

$$v_i = \max\{j : \underline{t}_{ij} \geq t\},\ w_i = \max\{j : \overline{t}_{ij} \leq t\}.$$

Reliability of a series system (Possibilistic data)

Independent units:

$$\underline{R}(t) = 1 - \prod_{i=1}^{n}(1 - \underline{p}_{iw_i}),\ \overline{R}(t) = 1 - \prod_{i=1}^{n} \underline{p}_{iv_i},$$

No information about independence:

$$\begin{aligned}\underline{R}^*(t) &= \max_{i=1,\dots,n} \underline{p}_{iw_i}, \\ \overline{R}^*(t) &= 1 - \max\left(0, \sum_{i=1}^{n} \underline{p}_{iv_i} - (n-1)\right).\end{aligned}$$

Similar expressions are obtained for other typical systems.

The obtained reliability measures only partially coincide with the corresponding measures obtained in the framework of the possibility theory!

Multi-state and continuum-state systems

The system units can be:

- in two states {work, fail}: a *binary* system;
- in m states from work till fail: a *multi-state* system;
- in real interval $[0, T]$ of states: a *continuum* system.

Utkin and Gurov 1999 *Imprecise reliability of general structures*
Utkin and Kozine 2002 *A reliability model of multi-state units under partial information*
Utkin 2006 *Cautious reliability analysis of multi-state and continuum-state systems based on the imprecise Dirichlet model*

Fault tree analysis

Fault tree analysis is a logical and diagrammatic method to evaluate the probability of an accident resulting from sequences and combinations of faults and failure events. Fault tree analysis can be regarded as a special case of event tree analysis

Cano and Moral 2002 *Using probability trees to compute marginals with imprecise probability*

Troffaes and Coolen 2007 *On the use of the imprecise Dirichlet model with fault trees*

Structural reliability (1)

The structural reliability problem in a general form:
$\Pr\{g(\widetilde{X}) > 0\} = ?$.
The most useful and simple special case: reliability
$R = \Pr\{Y - X > 0\} = ?$.
Y is the r.v. describing the strength of a system,
X is the r.v. describing the stress or load placed on the system.

Structural reliability (2)

Given n and m interval-valued points of CDFs of the stress and strength $\underline{p}_i \leq \Pr\{X \leq x_i\} \leq \overline{p}_i$, $\underline{q}_j \leq \Pr\{Y \leq y_j\} \leq \overline{q}_j$.

- Lack of information about independence

$$\begin{aligned}\underline{R}^* &= \max_{i=1,\dots,n} \max\left(0, \underline{p}_i - \overline{q}_{j(i)}\right), \ j(i) = \min\{j : x_i \leq y_j\}, \\ \overline{R}^* &= 1 - \max_{k=1,\dots,m} \max\left(0, \underline{q}_k - \overline{p}_{l(k)}\right), \ l(k) = \min\{l : y_k \leq x_l\}.\end{aligned}$$

- X and Y are independent

$$\underline{R} = \sum_{i=1}^{n}(\underline{p}_i - \underline{p}_{i-1})(1 - \overline{q}_{j(i)}), \ j(i) = \min\{j : x_i \leq y_j\},$$

$$\overline{R} = 1 - \sum_{k=1}^{m}(\underline{q}_k - \underline{q}_{k-1})(1 - \overline{p}_{l(k)}), \ l(k) = \min\{l : y_k \leq x_l\}.$$

Structural reliability (3)

Given $\underline{\mathbf{E}}X$, $\overline{\mathbf{E}}X$, $\underline{\mathbf{E}}Y$, $\overline{\mathbf{E}}Y$. T_X and T_Y are limited values of X and Y, respectively.

- Lack of information about independence

$$\underline{R}^* = \max\{0, ((\underline{\mathbf{E}}Y - \overline{\mathbf{E}}X)/T_Y)\},$$
$$\overline{R}^* = \min\{1, 1 - (\underline{\mathbf{E}}X - \overline{\mathbf{E}}Y)/T_X\}.$$

- X and Y are independent

$$\underline{R} = \max\{0, ((\underline{\mathbf{E}}Y - \overline{\mathbf{E}}X)/T_Y)\},$$
$$\overline{R} = \min\{1, 1 - (\underline{\mathbf{E}}X - \overline{\mathbf{E}}Y)/T_X\}.$$

Approaches for analyzing the repairable systems

1. Markov chains (Kozine and Utkin 2002, Campos, et al. 2008, Skulj 2006, de Cooman, Hermans, Quaeghebeur 2008, etc.)
   Assumptions: times to failure and repair are exponentially distributed.

Approaches for analyzing the repairable systems

1. Markov chains (Kozine and Utkin 2002, Campos, et al. 2008, Skulj 2006, de Cooman, Hermans, Quaeghebeur 2008, etc.)
   Assumptions: times to failure and repair are exponentially distributed.
2. Systems of integral equations:

$$\begin{cases} y_0(s,t) = \int_0^t f(x+s) y_1(0,t-x)\mathrm{d}x + f(t+s) \\ y_1(\tau,t) = \int_0^t g(x+\tau) y_0(0,t-x)\mathrm{d}x \end{cases}.$$

$$p_0(t) = \int_0^\infty y_0(s,t)\mathrm{d}s.$$

   Optimal solutions: $p_0(t) \longrightarrow \min_{w_i,x_i,v_i,z_i}(\max)$

$$f_o(x) = \sum w_i \delta(x - x_i),\ \ g_o(x) = \sum v_i \delta(x - z_i).$$

Imprecise software growth reliability models

This question is discussed at this symposium in detail.

Open Problems

- Human reliability
- Security engineering
- Different independence conditions and their “reliability” sense
- New efficient imprecise models of statistical inference
- Efficient methods for solving the optimization problems (natural extension)
- **and many many other problems whose solution in the framework of imprecise probabilities opens new perspectives in reliability!**

Questions

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