Sets of Desirable Gambles and Credal Sets

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ISIPTA 09 - Durham, U.K.

- Inés and myself: interest in Imprecise Probability in Departments with a large number of people working in Fuzzy Logic.
- Enrique Miranda is working in Inés department.
- In my department there is also a large group of people working in Bayesian graphical models.
- People with interest in IP: Joaquín Abellán (Classification) and Andrés Cano (computation in credal networks).
- We are in a Computer Science department and it is very difficult to find people with the appropriate background and interest to work in IP.

Walley, 2000

Sets of desirable gambles and partial preference orderings are the most informative of the mathematical models I have discussed, and they seem to be able to model all the common types of uncertainty. They uniquely determine upper and lower previsions and conditional previsions, and they contain all the information about preferences that is relevant in making decisions. In many ways they are the simplest and most natural mathematical models. The coherence axioms and rules of inference (natural extension) for sets of desirable gambles are especially simple. In this paper, I have advocated these models on the grounds of mathematical generality, but it is also arguable that they are the simplest and most natural models from the point of view of interpretation.

Walley's book. Page 295

Justification of the Coherence Axioms

.... The marginal gamble $G(X)$ is almost desirable, by interpretation of $P(X)$, and $G(Y|\mathcal{B})$ is almost desirable...Hence the sum $G(X) + G(Y|\mathcal{B})$ is almost desirable ....

Question

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Gambles

- We have an unkonwn value belonging to a finite set Ω
- A gamble is a mapping $X : \Omega \rightarrow \mathbb{R}$
- The value $X(\omega)$ is the reward (loss in the case of a negative value) we obtain under gamble X if ω is the true unknown value (in additive precise utilities)
- Some gambles are clearly desirable for us (for example if $X(\omega) > 0, \forall\omega$) and other are undesirable (for example if $X(\omega) < 0, \forall\omega$)

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Example

- Consider the result of football match with $\Omega = \{0-0, 1-0, 0-1, 1-1, 2-0, ..., 15-15\}$
- A gamble $X_1(1-0) = 10, X_1(r) = -1$, otherwise
- If we believe in that draw is going to be the result we could accept:

$$X_3(i-i) = 1, \quad X_3(i-j) = -1, i \neq j$$

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Desirable Gambles

Coherent Set of Desirable Gambles

D1. $0 \notin \mathcal{D}$,
D2. if $X \in \mathcal{L}$ and $X > 0$ then $X \in \mathcal{D}$,
D3. if $X \in \mathcal{D}$ and $c \in \mathbb{R}^+$ then $cX \in \mathcal{D}$,
D4. if $X \in \mathcal{D}$ and $Y \in \mathcal{D}$ then $X + Y \in \mathcal{D}$.

Basic Consistency Condition

A set of desirable gambles $\mathcal{D}$ avoids partial loss if and only if $0 \notin \mathcal{D}$

We should not accept a gamble in which $X(i - j) = -1$ if $i > j$ and 0 otherwise.

Almost Desirable Gambles

- D1' $\forall X \in \mathcal{D}^*$ such that $\sup X \geq 0$
- D2 If $X > 0$, then $X \in \mathcal{D}^*$
- D3 If $X \in \mathcal{D}^*$ and $\lambda > 0$ then $\lambda . X \in \mathcal{D}^*$
- D4 If $X_1, X_2 \in \mathcal{D}^*$ then $X_1 + X_2 \in \mathcal{D}^*$
- D5 If $X + \epsilon \in \mathcal{D}^*, \quad \forall \epsilon > 0$ then $X \in \mathcal{D}^*$

Basic Consistency Condition

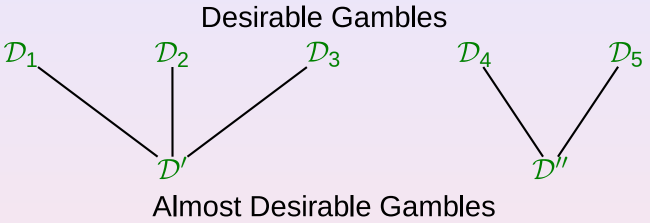
A set of almost desirable gambles $\mathcal{D}^*$ avoids sure loss if and only if $\forall X \in \mathcal{D}$ such that $\sup X \geq 0$

Desirable vs. almost desirable gambles

Let us consider the gambles:
$X_\epsilon(i-j) = \epsilon$ if $i-j \neq 15-15$
$X_\epsilon(15-15) = -1$

- It is possible that all the gambles X_ϵ are desirable (and almost desirable).
- If they are almost desirable, then the gamble:
  $X_0(i-j) = 0$ if $i-j \neq 15-15$
  $X_0(15-15) = -1$
  is almost desirable.
- The set of almost desirable gambles avoids sure loss, but it does not avoid partial loss.

Desirable vs Almost Desirable Gambles



Desirable Gambles are a more general model

Strictly Desirable Gambles

D2 If $X > 0$, then $\mathcal{D}$

D3 If $X \in \mathcal{D}$ and $\lambda > 0$ then $\lambda.X \in \mathcal{D}$

D4 If $X_1, X_2 \in \mathcal{D}$ then $X_1 + X_2 \in \mathcal{D}$

D5' If $X \in \mathcal{D}$ then either $X > 0$ or $\exists \epsilon > 0,\ X - \epsilon \in \mathcal{D}$

Credal Sets and Desirable Gambles

- A set of desirable gambles $\mathcal{D}$ defines a credal set (a closed and convex set of probability measures):

$$\mathcal{K} = \{P \,:\, P[X] \geq 0, \forall X \in \mathcal{D}\}$$

- A set of desirable gambles $\mathcal{D}$ and its associated set of almost desirable gambles $\mathcal{D}^*$ define the same credal set
- A credal set $\mathcal{K}$ defines a set of almost desirable gambles:

$$\mathcal{D} = \{X \,:\, P[X] \geq 0, \forall P \in \mathcal{K}\}$$

- But several sets of desirable gambles can be associated:

$$\mathcal{D}' = \{X \,:\, P[X] > 0, \forall P \in \mathcal{K}\} \cup \{X \,:\, X > 0\}$$

$$\mathcal{D}'' = \{X \,:\, P[X] \geq 0, \forall P \in \mathcal{K}, \quad \exists P \in \mathcal{K},\ P[X] > 0\} \cup \{X \,:\, X > 0\}$$

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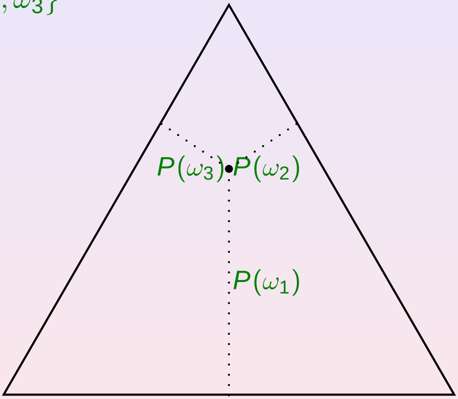
Graphical Representation: Probability Distribution

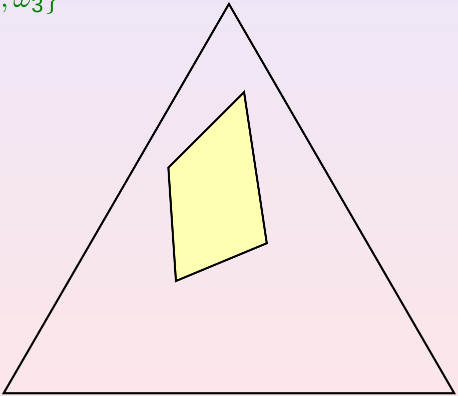
$\Omega = \{\omega_1, \omega_2, \omega_3\}$



Graphical Representation: Credal Set

$$E_P[X] \geq 0, \forall P \in \mathcal{K}$$

$\Omega = \{\omega_1, \omega_2, \omega_3\}$



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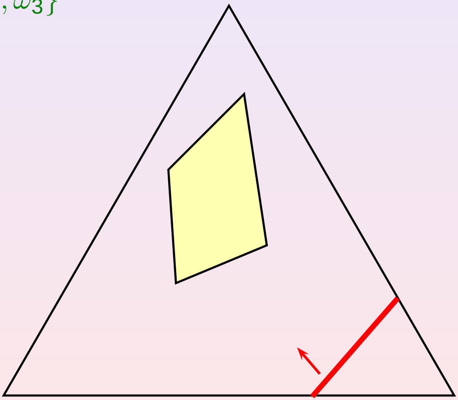

Desirable Gamble

Graphical Representation: Credal Set

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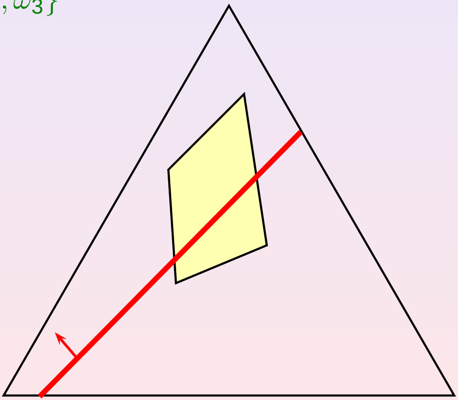

Non Desirable Gamble

Graphical Representation: Credal Set

$$E_P[X] \geq 0, \forall P \in \mathcal{K}$$

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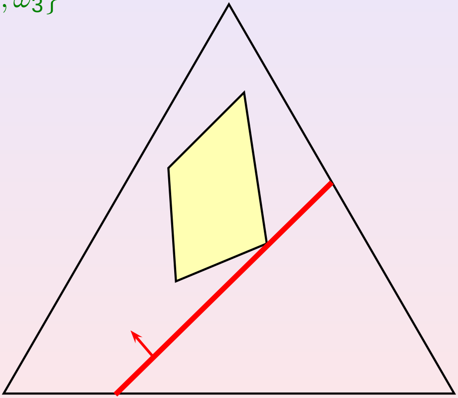

Almost Desirable Gamble, Desirable? Non necessary, but possible

Conditioning

If we have a set of desirable gambles $\mathcal{D}$ and we observe event B, the conditional set of desirable gambles given B is given by:

$$\mathcal{D}_{\mathcal{B}} = \{X \ : \ X.B \in \mathcal{D}\} \cup \{X \ : \ X > 0\}$$

Example

I we accept a gamble
$X(Win) = 1, \quad X(Loss) = -1, \quad X(Draw) = 0,$
if we know that *Draw* has not happened, then we should accept any gamble:
$Y(Win) = 1, \quad Y(Loss) = -1, \quad Y(Draw) = \alpha$

All the conditional information is in $\mathcal{D}$.

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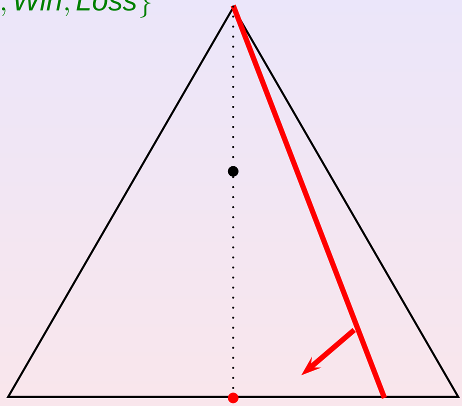
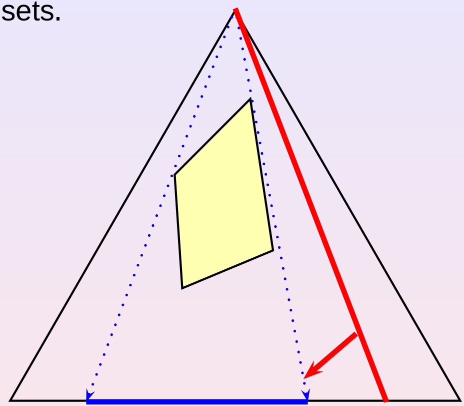
$\Omega = \{Draw, Win, Loss\}$



$B = \{Win, Loss\}$

Graphical Representation: Credal Set

$\Omega = \{Draw, Win, Loss\}$. Conditioning in terms of desirability and credal sets.

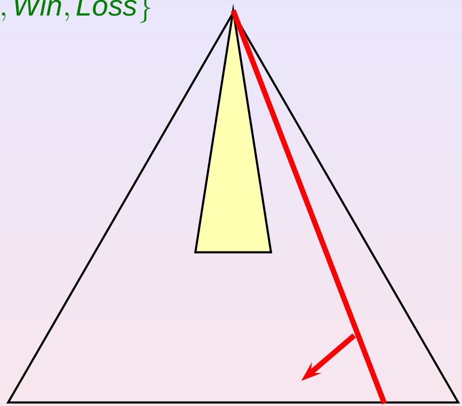


$B = \{Win, Loss\}$

If $\underline{P}(B) > 0$, then the credal set associated to the conditional set $\mathcal{D}$ is uniquely determined with independence of what happens with gambles in the frontier.

Conditioning: Lower Probability equal to 0

$\Omega = \{Draw, Win, Loss\}$



$B = \{Win, Loss\}$

If $\underline{P}(B) = 0$, all the gambles with $X(D) = 0.0$ are in the frontier.
The credal set does not contains information about the conditioning.

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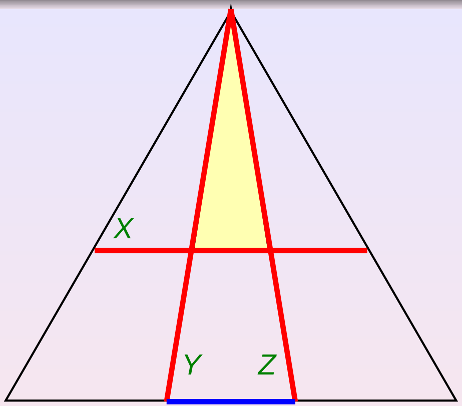

$B = \{Win, Loss\}$

This situation is compatible with accepting as desirable the gambles:

$$X(D) = 1, \quad X(W) = -1, \quad X(L) = -1$$
$$Y(D) = 0, \quad Y(W) = 1.2, \quad Y(L) = -1$$
$$Z(D) = 0, \quad Z(W) = -1, \quad Z(L) = 1.2$$

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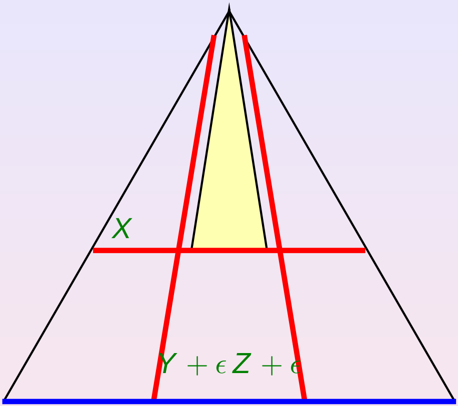

$B = \{Win, Loss\}$

But it is also compatible with gambles $\{X, Y + \epsilon, Z + \epsilon\}$

In this case, the conditioning is very wide: There is no desirable non-trivial gamble T with $T(D) = 0$.

Regular Extension

- I have an urn with *Red*, *Blue*, *White* balls.
- I know that there is exactly the same number of Blue and White balls.
- This situation can be represented by the convex set of probability distributions:

	Red	*Blue*	*White*
P_1	1	0	0
P_2	0	0.5	0.5

- If the set of desirable gambles is:

$$\mathcal{D}' = \{X \ : \ P[X] > 0, \forall P \in \mathcal{P}\} \cup \{X \ : \ X > 0\}$$

  then, if we know that a ball randomly selected from the urn is not red, then conditional to this information, the gamble $X(Blue) = 2, \quad X(White) = -1$ is not accepted.
- This does not seem reasonable. I should accept any gamble in which $X(Blue) + X(White) > 0$.
- This is obtained under regular extension:

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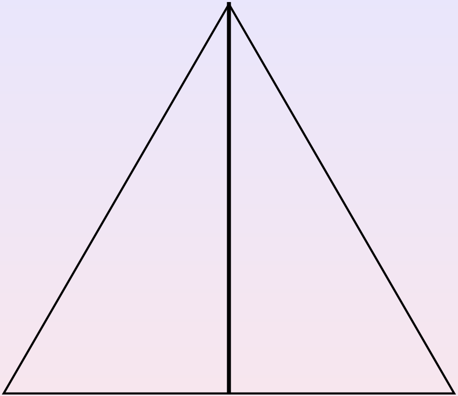
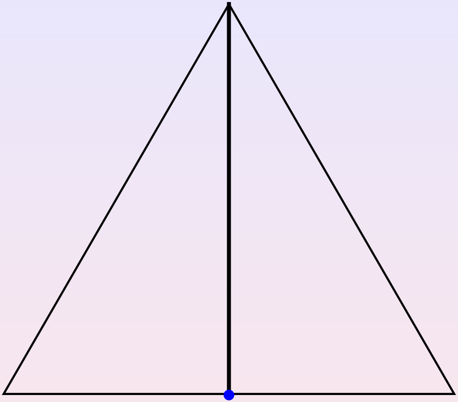
Geometric Representation

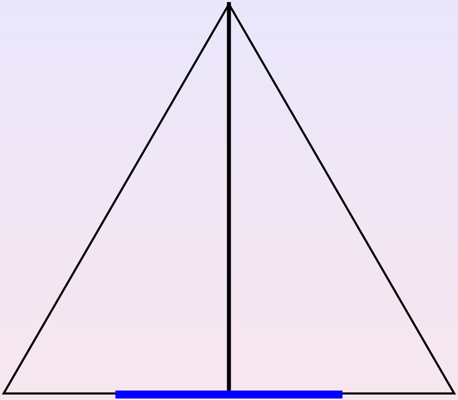


Geometric Representation

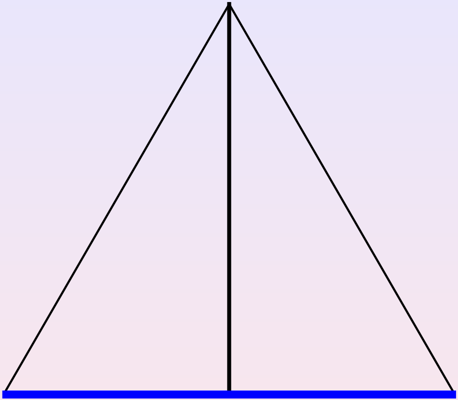


This corresponds to the case of the urns Regular Extension

Geometric Representation



One possible option for the betting on results



The least informative option Natural Extension

Regular Extension

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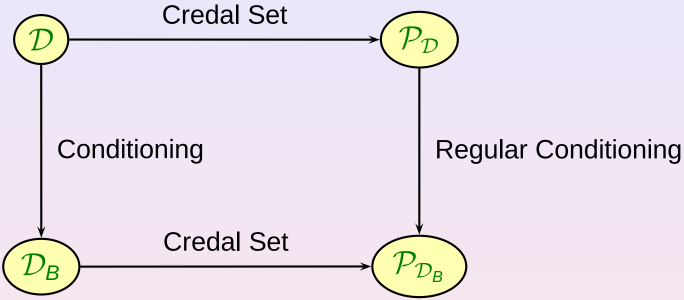
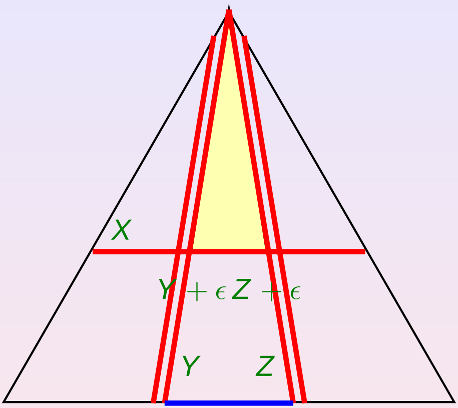


Theorem

Desirable gambles, regular extension is obtained assuming:

$$X \in \mathcal{D}^* \text{ and } -X \notin \mathcal{D}^* \Rightarrow X \in \mathcal{D}.$$

Conditioning: Regular Extension



$B = \{Win, Loss\}$

Y and Z are the limit of desirable gambles, and the opposite gambles are not almost desirable, and then they are desirable.

Other Questions to be Discussed (poster)

- Natural extension
- Representing gambles
- Algorithms for checking coherence and avoiding partial loss
- Maximal coherent set of gambles: sequences of probability measures.