

# Exchangeability for sets of desirable gambles

Gert de Cooman & Erik Quaeghebeur



Gert  
de Cooman



Erik  
Quaeghebeur



## SYSTeMS



Filip  
Hermans



Keivan  
Shariatmadar



Gert  
de Cooman



Erik  
Quaeghebeur



SYSTeMS



FUM



Filip  
Hermans



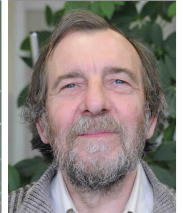
Keivan  
Shariatmadar



Gert  
de Cooman



Erik  
Quaeghebeur



Etienne  
Kerre

# Exchangeability for sets of desirable gambles

Gert de Cooman & Erik Quaeghebeur





# Immediate prediction under exchangeability . . .

14 International Conference on Approximate Probability Theory and Applications, Prague, Czech Republic, 2007

**Immediate prediction under exchangeability and representation insensitivity**

Gert de Cooman, Enrique Miranda, Erik Quaeghebeur  
Gert de Cooman, Enrique Miranda, Erik Quaeghebeur  
 Ghent University, Ghent, Belgium

**Immediate prediction under exchangeability & representation insensitivity**  
 Gert de Cooman, Enrique Miranda & Erik Quaeghebeur

**1 The setting**

**2 Requirements**

**3 Representation insensitivity**

**4 Immediate prediction**

**5 Conclusions**

**6 Acknowledgements**

Department of Statistics and Operations Research  
 Rey Juan Carlos University  
 Calle Turis, s/n 28002 Madrid, Spain

**Author's personal copy**

International Journal of Approximate Reasoning 70 (2016) 256–262

Contents lists available at ScienceDirect

**International Journal of Approximate Reasoning**

ELSEVIER

journal homepage: www.elsevier.com/locate/ijar

**Representation insensitivity in immediate prediction under exchangeability<sup>a</sup>**

Gert de Cooman<sup>a,\*</sup>, Enrique Miranda<sup>b</sup>, Erik Quaeghebeur<sup>a,1</sup>

<sup>a</sup> Ghent University, IMBIM Research Group, Technologiepark 70, 9000 Ghent, Belgium  
<sup>b</sup> Rey Juan Carlos University, Department of Statistics, s/n 2, Turis, s/n, 28002 Madrid, Spain

**ARTICLE INFO**

**ABSTRACT**

**1. Introduction**

Consider a subject who is making  $N > 0$  successive observations of a certain phenomenon. We represent these observations by  $N$  random variables  $X_1, \dots, X_N$ . The random variable  $X_n$  represents a variable whose value the subject may estimate and predict. We assume that at each observation instant  $n$ , the actual value of the random variable  $X_n$  can be determined in principle. To be able, one subject might be looking for bugs in the Russian forest, and then it is the operator of the lifting for cancer screens. Or, he might, as an ecological manager, be tracking both wetland populations from one site, as which case  $X_n$  could designate the color of the UK ball labels from the urn.

**2. Overview**

This paper has been partially supported by the research Grant G42013 of the Flemish Fund for Scientific Research (FWO) and by the projects PROMETHEUS, PROMETHEUS2 and PROMETHEUS3 (G. de Cooman).

\* Corresponding author.  
 E-mail addresses: gert.decooman@ugent.be (G. de Cooman), erik.quaeghebeur@ugent.be (E. Quaeghebeur).

Research funded by a Ph.D. Grant of the Institute for the Promotion of Innovation through Scientific and Technological Research (PROMETHEUS).

© 2016 Elsevier Inc. All rights reserved.



# Immediate prediction under exchangeability . . .

- ▶ families of exchangeable lower previsions
- ▶ coherent updated exchangeable lower previsions
- ▶ count vectors as sufficient statistics

18 International Association for Applied Probability - Theoretical and Applications, Prague, Czech Republic, 2007

**Immediate prediction under exchangeability and representation insensitivity**

Gert de Cooman  
Erik Miranda  
Erik Quaghebeur

**Immediate prediction under exchangeability & representation insensitivity**

Gert de Cooman, Enrique Miranda & Erik Quaghebeur

Department of Statistics and Operations Research  
Ryerson University  
Toronto, Canada

**1 The setting**

Let  $\Omega$  be a finite set. A random variable  $X$  on  $\Omega$  is a function  $X: \Omega \rightarrow \mathcal{X}$ . A random variable  $X$  is said to be exchangeable if for any permutation  $\pi$  of  $\Omega$ , the joint distribution of  $(X_1, \dots, X_n)$  is the same as that of  $(X_{\pi(1)}, \dots, X_{\pi(n)})$ .

**2 Representation insensitivity**

A lower prevision  $\mathbb{P}_X$  on  $\mathcal{X}$  is said to be representation insensitive if for any two lotteries  $L$  and  $L'$  such that  $L \sim L'$ , we have  $\mathbb{P}_X(L) = \mathbb{P}_X(L')$ .

**Author's personal copy**

International Journal of Approximate Reasoning 76 (2016) 284–310

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

Representation insensitivity in immediate prediction under exchangeability<sup>a</sup>

Gert de Cooman<sup>a,\*</sup>, Enrique Miranda<sup>a</sup>, Erik Quaghebeur<sup>a,1</sup>

**ARTICLE INFO**

**ABSTRACT**

**1. Introduction**

Consider a subject who is making  $N > 0$  successive observations of a certain phenomenon. We represent these observations by  $N$  random variables  $X_1, \dots, X_N$ . The random variable  $X_i$  receives a value in the random set  $\mathcal{X}_i$  and the domain of the probability is  $\mathcal{X}_i$ . We assume that at each observation instant  $i$ , the actual value of the random variable  $X_i$  is not determined in principle. To be able, one might wish to handle the steps in the process forward, and then to be the opposite of the following for some reason. It is, for example, an ecological experiment, for instance both without replacement (see, for example, [1, 2], could designate the order of the fish and labels from the sea).

**ARTICLE INFO**

Available online 14 June 2016

**ABSTRACT**

Representation insensitivity in immediate prediction under exchangeability is studied. It is shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions.

**ARTICLE INFO**

**ABSTRACT**

Representation insensitivity in immediate prediction under exchangeability is studied. It is shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions.

**ARTICLE INFO**

Available online 14 June 2016

**ABSTRACT**

Representation insensitivity in immediate prediction under exchangeability is studied. It is shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions.

**ARTICLE INFO**

**ABSTRACT**

Representation insensitivity in immediate prediction under exchangeability is studied. It is shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions. It is also shown that representation insensitivity is a natural property of exchangeable lower previsions.



# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\Omega$  of outcomes of some experiment.

A subject sets a valuation about the experiment's outcome.

Identifies  $f: \Omega \rightarrow \mathbb{R}^n$ , interpreted as gambler's results.

$f(\omega)$  is the gambler's outcome in  $\omega$ .



A gambler  $G$  is admitted to bet subject if he accepts the following (interchangeable) bet:

(I) the actual outcome  $\omega$  is determined, and

(II) the subject's capital is changed by  $f(\omega)$ .

The new gambler  $G'$  is not admitted.

## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{D} \subseteq \mathcal{F}(\Omega)$  models his beliefs about the experiment's outcome.

The set of desirable gambles of  $G$  is coherent

if it satisfies the following stability requirements:

(D1)  $0 \in \mathcal{D}$  (no bet).

(D2)  $f \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$  (adding gambles).

(D3)  $f \in \mathcal{D} \Rightarrow f \circ \pi \in \mathcal{D}$  (betting).

(D4)  $f, g \in \mathcal{D} \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}$  (betting).

Requirements D2 and D4 make  $\mathcal{D}$  a convex cone in  $\mathcal{F}(\Omega)$ .



## Sets of weakly desirable gambles

The subject considers a gamble  $f \in \mathcal{F}(\Omega)$  weakly desirable if he is willing to exchange gambles  $f$  & another desirable gamble  $g$ .

The subject's set of weakly desirable gambles is

$$\mathcal{D}_w := \{f \in \mathcal{F}(\Omega) \mid f \preceq g \circ \pi\}$$

The set of weakly desirable gambles  $\mathcal{D}_w$  corresponding to a coherent set of desirable gambles  $\mathcal{D}$  satisfies the following properties:

WDD1:  $f \in \mathcal{D}_w \Rightarrow f \circ \pi \in \mathcal{D}$  (betting).

WDD2:  $f \in \mathcal{D}_w \Rightarrow f + g \in \mathcal{D}_w$  (adding gambles).

WDD3:  $f \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD4:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD5:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD6:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD7:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD8:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD9:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD10:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD11:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD12:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD13:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD14:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD15:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD16:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD17:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD18:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD19:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD20:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD21:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD22:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD23:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD24:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD25:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD26:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD27:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD28:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD29:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD30:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD31:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD32:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD33:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD34:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD35:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD36:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD37:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD38:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD39:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD40:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD41:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD42:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD43:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD44:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD45:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD46:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD47:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD48:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD49:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD50:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD51:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD52:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD53:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD54:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD55:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD56:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD57:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD58:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD59:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD60:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD61:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD62:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD63:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD64:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD65:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD66:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD67:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD68:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD69:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD70:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD71:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD72:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD73:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD74:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD75:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

WDD76:  $f, g \in \mathcal{D}_w \Rightarrow f \circ \pi \vee g \circ \pi \in \mathcal{D}_w$  (betting).

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \Omega$ .

Contingent on observing  $B$ , the subject models his beliefs using an updated set of desirable gambles, the subject's  $\mathcal{D}^B$  given by

$$\mathcal{D}^B := \{f \in \mathcal{F}(\Omega) \mid f \circ \pi \in \mathcal{D}\}$$

If  $B$  is a coherent set of desirable gambles on  $\Omega$ , then  $\mathcal{D}^B$  is a coherent set of desirable gambles on  $\Omega$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  (associated to a set of desirable gambles  $\mathcal{D}$ ) is

$$\underline{P}(f) := \inf\{p \in \mathbb{R} \mid p - pf \in \mathcal{D}\}$$

Its conjugate upper prevision  $\overline{P}(f)$  is

$$\overline{P}(f) := \sup\{p \in \mathbb{R} \mid pf - p \in \mathcal{D}\}$$

A lower prevision  $\underline{P}$  is coherent if there exists some coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{P}(f) = \inf\{p \in \mathbb{R} \mid p - pf \in \mathcal{D}\}$ .

Coherent lower previsions are best representable uncertainty models, their coherent sets of desirable gambles.



Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a sequence  $X_1, \dots, X_n$  of random variables to which  $\Omega$  is the finite set of possible outcomes. So the possibility

space is  $\Omega = \mathcal{R}^n$  and  $\omega = (x_1, \dots, x_n)$ .

$f$  is a list of  $n$  elements,  $f = (f_1, \dots, f_n)$ .

$\mathcal{D}$  is the set of all permutations of the index set  $\{1, \dots, n\}$ .

The associated permutation  $\pi$  is defined by  $(\omega) \circ \pi = \omega_{\pi^{-1}(i)}$ .

It is fixed a permutation  $\sigma$  of  $\{1, \dots, n\}$  by letting  $f \circ \sigma = f \circ \pi$ .

With every sequence of observations corresponds a joint vector in  $\mathcal{R}^n$ ,  $\omega = (x_1, \dots, x_n)$ .

The identification  $\mathcal{R}^n \cong \mathcal{R}^n$  maps  $\mathcal{D}^*$  to  $\mathcal{D}$ .

The identification  $\mathcal{R}^n \cong \mathcal{R}^n$  maps  $\mathcal{D}^*$  to  $\mathcal{D}$ .

The identification  $\mathcal{R}^n \cong \mathcal{R}^n$  maps  $\mathcal{D}^*$  to  $\mathcal{D}$ .

Permitted gambles have the same joint vector: a permutation element  $g$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

$g = (x_1, \dots, x_n)$ .

## Exchangeability

A subject considers that  $X_1, \dots, X_n$  are exchangeable. This means that for any gamble  $f$  and any permutation  $\sigma$ , he finds exchanging  $f$  by  $f \circ \sigma$  weakly desirable, because he is indifferent between them.

# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\Omega$  of outcomes of some experiment.  
A subject sets a valuation about the experiment's outcome.  
Identifies  $f \in \mathcal{F}(\Omega) := \mathbb{R}^\Omega$  interpreted as gambles (results:  $f(\omega)$  when the experiment's outcome is  $\omega$ ).



A gamble  $f$  is desirable to the subject if it satisfies the following three properties:  
(1) Its actual outcome is  $\geq 0$ , and  
(2) Its subject's value is  $\geq 0$  (i.e.,  $f(\omega) \geq 0$ ).  
The same game  $f$  is not desirable.

## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{D} \subseteq \mathcal{F}(\Omega)$  models his beliefs about the experiment's outcome.  
The set of desirable gambles  $\mathcal{D}$  is coherent if it satisfies the following rationality requirements: (1)  $\mathcal{D} \cap \mathcal{F}(\Omega) = \emptyset$ , (2)  $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$ , (3)  $f \in \mathcal{D} \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}$  for all  $\lambda \geq 0$ , (4)  $f \in \mathcal{D} \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}$  for all  $\lambda \geq 0$ , (5)  $f \in \mathcal{D} \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}$  for all  $\lambda \geq 0$ .



## Sets of weakly desirable gambles

The subject considers a gamble  $f \in \mathcal{F}(\Omega)$  weakly desirable if he identifies any desirable gamble  $g$  as another desirable gamble  $h$  observed, so  $f \in \mathcal{D}$  if then  $f + g \in \mathcal{D}$ .

The subject's set of weakly desirable gambles is  $\mathcal{D}_w := \{f \in \mathcal{F}(\Omega) : f + g \in \mathcal{D}\}$ .  
The set of weakly desirable gambles  $\mathcal{D}_w$  corresponding to a coherent set of desirable gambles  $\mathcal{D}$  satisfies the following properties:  
(1)  $f, g \in \mathcal{D}_w \Rightarrow f + g \in \mathcal{D}_w$ ,  
(2)  $f \in \mathcal{D}_w \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}_w$  for all  $\lambda \geq 0$ ,  
(3)  $f \in \mathcal{D}_w \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}_w$  for all  $\lambda \geq 0$ ,  
(4)  $f \in \mathcal{D}_w \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}_w$  for all  $\lambda \geq 0$ .



## Assessments & their natural extension

An assessment can consist of a set  $\mathcal{A} \subseteq \mathcal{F}(\Omega)$  considered desirable by the subject.  
The assessment of a gamble  $f$  under non-possibility of the reduction of  $\text{con}(\mathcal{A})$  and  $\mathcal{K}_c(\mathcal{A})$  is empty.  
The natural extension  $\mathcal{D}^e$  of  $\mathcal{A}$  is  $\mathcal{D}^e := \text{con}(\mathcal{A}) \cup \mathcal{K}_c(\mathcal{A})$ .



If  $\mathcal{A}$  of gambles non-possibility, then  $\mathcal{D}^e(\mathcal{A})$  is the smallest coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \Omega$ .  
Contingent on observing  $B$ , the subject models his beliefs using an updated set of desirable gambles, the subject of  $\mathcal{D}^B$  given by  $\mathcal{D}^B := \{f \in \mathcal{F}(\Omega) : f + g \in \mathcal{D}\}$ .

If  $\mathcal{D}$  is a coherent set of desirable gambles on  $\Omega$ , then  $\mathcal{D}^B$  is a coherent set of desirable gambles on  $B$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  established to a set of desirable gambles  $\mathcal{D}$  is  $\underline{P}(f) := \inf\{x \in \mathbb{R} : f - x \mathbf{1} \in \mathcal{D}\}$ .

Its conjugate upper prevision  $\overline{P}(f)$  is  $\overline{P}(f) := \sup\{x \in \mathbb{R} : x - f \in \mathcal{D}\}$ .

A lower prevision  $\underline{P}$  is coherent if there exists some coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} : f - x \mathbf{1} \in \mathcal{D}\}$ .

Coherent lower previsions are best representable uncertainty models, their coherent sets of desirable gambles.



Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a sequence  $\omega_1, \dots, \omega_n$  of random variables for which  $\Omega$  is the finite set of possible values. So the possibility space is  $\mathcal{F} := \{f : \Omega \rightarrow \mathbb{R}\}$ .  
Upper  $\mathcal{D}^u$  and lower  $\mathcal{D}^l$  are  $\mathcal{D}^u := \{f \in \mathcal{F} : f(\omega) \geq 0\}$  and  $\mathcal{D}^l := \{f \in \mathcal{F} : f(\omega) \leq 0\}$ .  
 $\mathcal{D}^u$  is the set of all permutations  $\pi$  of the sides set  $\{1, \dots, n\}$ .  
The associated permutation  $\pi^*$  is defined by  $\pi^*(\omega) = \omega_{\pi(i)}$ .  
It is called a permutation of  $\mathcal{F}(\Omega, \mathcal{F}^*)$  by letting  $\mathcal{D}^u$  or  $\mathcal{D}^l$  in  $\mathcal{F}^*$ .

With every sequence of observations corresponding a joint vector in  $\mathbb{R}^n := \mathbb{R} \times \dots \times \mathbb{R}$ .  
The identification  $\mathbb{R}^n \cong \mathbb{R}^n$  maps  $\mathcal{F}^*$  into  $\mathbb{R}^n$ .  
A permutation  $\pi$  is a vector  $\pi = (\pi_1, \dots, \pi_n) \in \mathbb{R}^n$ .

Permitted gambles have the same joint vector: a permutation invariant gamble  $g$  is  $g(\omega) = g(\pi(\omega))$ .

$g(\omega) = g(\pi(\omega))$  is  $g(\omega) = g(\pi(\omega))$ .

## Exchangeability

A subject assesses that  $\omega_1, \dots, \omega_n$  are exchangeable. This means that for any gamble  $f$  and permutation  $\pi$ , he finds exchangeable  $f$  for  $\pi f$  for  $\pi$  desirable, because he is indifferent between them.  
The negative independence of each of such exchange gambles is  $\mathcal{D}_w := \{f - \pi f : f \in \mathcal{D}_w\}$  and  $\mathcal{K}_c(\mathcal{D}_w)$ .

A  $\mathcal{D}_w$  is a set of weakly desirable gambles, then so does its conical hull  $\text{con}(\mathcal{D}_w) = \mathcal{D}_w^+$  or  $\text{span}(\mathcal{D}_w)$ .

A subset  $\mathcal{A}$  of all assessment gambles on  $\mathbb{R}^n$  is called exchangeable if  $\mathcal{A} \subseteq \mathcal{D}_w^+$  or, equivalently, if  $\mathcal{A} \subseteq \mathcal{D}_w^+$ .

If  $\mathcal{A}$  is coherent and exchangeable then it is also permutation for all  $\pi$  if and only if  $\mathcal{A} \subseteq \mathcal{D}_w^+$  holds for  $\mathcal{F}(\mathbb{R}^n)$ .

## Exchangeable natural extension

The assessment of a gamble  $f$  non-possibility under exchangeability of  $\mathcal{A}$  is  $\mathcal{D}_w^+(\mathcal{A})$  avoids non-possibility.

The exchangeable natural extension of  $\mathcal{A}$  is  $\mathcal{D}_w^+(\mathcal{A}) := \text{con}(\mathcal{A}) \cup \mathcal{K}_c(\mathcal{A})$ .

If  $\mathcal{A}$  is a coherent non-possibility under exchangeability then  $\mathcal{D}_w^+(\mathcal{A})$  is the smallest exchangeable coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating exchangeable models

The subject observes the values  $\omega_1, \dots, \omega_n$  at the joint vector  $\omega \in \mathbb{R}^n$  or the first  $n$  variables  $\omega_1, \dots, \omega_n$  the event observing the event  $\{i : \omega_i = 1\} \subseteq \mathbb{R}^n$ . We are interested in observations about the relationship  $\omega_i = 1$  or  $\omega_i = 0$ .

Contingent on observing  $i$  or  $\omega_i$ , the subject models his beliefs using updated sets of desirable gambles, the subsets of  $\mathcal{D}_w^+$  that are  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 1)$  and  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 0)$ .

If  $\mathcal{A}$  is a coherent and exchangeable set of desirable gambles on  $\mathbb{R}^n$ , then  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 1)$  and  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 0)$  are coherent and exchangeable sets of desirable gambles on  $\mathbb{R}^{n-1}$ .

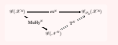
On non-exchangeable joint vectors are sufficient statistics:  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 1)$  and  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 0)$ .

## Exchangeable previsions

The values  $\underline{P}(\mathcal{A} | \omega_i = 1)$  is exchangeable if there is some non-exchangeable subset  $\mathcal{A}$  of desirable gambles of such that  $\underline{P}(\mathcal{A} | \omega_i = 1) = \underline{P}(\mathcal{A} | \omega_i = 0)$ .

Representation

## Moving between sequence gambles and count gambles



The set of permutation invariant sequence gambles is  $\mathcal{F}(\Omega, \mathcal{F}^*) := \{f \in \mathcal{F}(\Omega, \mathcal{F}^*) : f(\omega) = f(\pi(\omega))\}$ .

The projection of a sequence gamble  $f$  onto a permutation invariant sequence gamble is  $\text{con}(\mathcal{F}(\Omega, \mathcal{F}^*)) := \text{con}(\mathcal{F}(\Omega, \mathcal{F}^*))$ .

The count gamble corresponding to the sequence gamble  $f$  is  $\text{con}(\mathcal{F}(\Omega, \mathcal{F}^*)) := \text{con}(\mathcal{F}(\Omega, \mathcal{F}^*))$ .

The permutation invariant sequence gamble  $f$  is a one-to-one correspondence with the count gamble  $g$  is  $f(\omega) = g(\pi(\omega))$ .

## Representation

A set of desirable gambles  $\mathcal{A}$  on  $\mathbb{R}^n$  is coherent and exchangeable if there is some coherent set  $\mathcal{D}$  of desirable gambles on  $\mathbb{R}^n$  - its natural representation - such that  $\mathcal{A} = \text{con}(\mathcal{D}, \mathcal{F}(\mathbb{R}^n))$ .

and in that case this  $\mathcal{D}$  is uniquely determined by  $\mathcal{D} = \{f \in \mathcal{F}(\mathbb{R}^n) : f(\omega) \geq 0\}$ .

$\mathcal{D} = \{f \in \mathcal{F}(\mathbb{R}^n) : f(\omega) \geq 0\}$ .

## Exchangeable natural extension & representation

The assessment of  $\mathcal{A}$  is  $\mathcal{D}_w^+(\mathcal{A})$  avoids non-possibility under exchangeability if  $\mathcal{D}_w^+(\mathcal{A})$  avoids non-possibility.

A natural representation  $\mathcal{D}_w^+(\mathcal{A}) := \text{con}(\mathcal{A}) \cup \mathcal{K}_c(\mathcal{A})$ .

$\mathcal{D}_w^+(\mathcal{A}) := \text{con}(\mathcal{A}) \cup \mathcal{K}_c(\mathcal{A})$ .

$\mathcal{D}_w^+(\mathcal{A}) := \text{con}(\mathcal{A}) \cup \mathcal{K}_c(\mathcal{A})$ .

## Representing updated models

The subject observes the values  $\omega_1, \dots, \omega_n$  at the joint vector  $\omega \in \mathbb{R}^n$  or the first  $n$  variables  $\omega_1, \dots, \omega_n$ .

If  $\mathcal{A}$  is a coherent and exchangeable set of desirable gambles on  $\mathbb{R}^n$ , then its representation of the lower - because of symmetry - identical updated models to use is  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 1)$  and  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 0)$ .

This representation is not an updated model of the representation  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 1)$  of  $\mathcal{A}$ . They are however related by  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 1) = \mathcal{D}_w^+(\mathcal{A} | \omega_i = 0) + \mathbf{1}_i$ .

when we use the identical function, defined for every count vector  $\omega \in \mathbb{R}^{n-1}$  by  $\mathcal{D}_w^+(\mathcal{A} | \omega_i = 1) = \mathcal{D}_w^+(\mathcal{A} | \omega_i = 0) + \mathbf{1}_i$ .

which is zero when  $j \neq i$ .

## Exchangeable previsions & representation

A lower prevision  $\underline{P}$  on  $\mathcal{F}(\mathbb{R}^n)$  is coherent and exchangeable if there is some coherent lower prevision  $\underline{P}$  on  $\mathcal{F}(\mathbb{R}^n)$  - its natural representation - such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} : f - x \mathbf{1} \in \mathcal{D}_w^+(\mathcal{A})\}$ .

- ▶ exchangeability assessment & weakly desirable gambles
- ▶ exchangeable natural extension
- ▶ updating exchangeable models
- ▶ relationship with exchangeable previsions

# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\Omega$  of outcomes of some experiment.  
A subject sets a valuation about the experiment's outcome.  
Standard  $f: \Omega \rightarrow \mathbb{R}^n$  interpreted as gambles means:  
 $f(\omega)$  is the experiment's outcome is  $\omega$ .



## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{D} \subseteq \mathbb{R}^n$  models his beliefs about the experiment's outcome.  
The set of desirable gambles  $\mathcal{D}$  is coherent if it satisfies the following rationality requirements:  
D1.  $0 \in \mathcal{D}$  if and only if  $f \in \mathcal{D}$ .  
D2.  $f \in \mathcal{D}$  implies  $f + g \in \mathcal{D}$  for any gamble  $g$ .  
D3.  $f \in \mathcal{D}$  implies  $f + \lambda \cdot 1 \in \mathcal{D}$  for any  $\lambda > 0$ .  
D4.  $f, g \in \mathcal{D}$  implies  $f + \lambda \cdot 1 \in \mathcal{D}$  for any  $\lambda > 0$ .  
Requirements D2 and D3 make  $\mathcal{D}$  a convex cone in  $\mathbb{R}^n$ .



## Sets of weakly desirable gambles

The subject considers a gamble  $f \in \mathbb{R}^n$  weakly desirable if he rejects any desirable gamble  $g$  & another desirable gamble  $h$  observed, so  $f \in \mathcal{D}$  if then  $f + g \in \mathcal{D}$ .  
The subject's set of weakly desirable gambles is:  
 $\mathcal{D}_w = \{f \in \mathbb{R}^n \mid f + g \in \mathcal{D}\}$ .



## Assessments & their natural extension

An assessment can consist of a set  $\mathcal{A} \subseteq \mathbb{R}^n$  (considered desirable) by the subject.  
The assessment is called non-desirable if the intersection of  $\text{conv}(\mathcal{A})$  and  $\mathcal{D}_w$  is empty.  
The natural extension of  $\mathcal{A}$  is:  
 $\mathcal{D}(\mathcal{A}) := \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$ .



If  $\mathcal{A}$  is acyclic non-desirable, then  $\mathcal{D}(\mathcal{A})$  is the smallest coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \Omega$ .  
Contingent on observing  $B$ , the subject models his beliefs using an updated set of desirable gambles. The subject's  $\mathcal{D}(B)$  given by:  
 $\mathcal{D}(B) = \{g \in \mathbb{R}^n \mid g \in \mathcal{D}\}$ .

If  $\mathcal{A}$  is a coherent set of desirable gambles on  $\Omega$ , then  $\mathcal{D}(B)$  is a coherent set of desirable gambles on  $B$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  established by a set of desirable gambles  $\mathcal{D}$  is:  
 $\underline{E}(f) := \max\{c \in \mathbb{R} \mid f - c \cdot 1 \in \mathcal{D}\}$ .

Its conjugate upper prevision  $\overline{E}(f)$  is equal to  $-\underline{E}(-f)$ .

A lower prevision  $\underline{E}$  is coherent if there exists a coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{E}(f) = \max\{c \in \mathbb{R} \mid f - c \cdot 1 \in \mathcal{D}\}$ .

Coherent lower previsions are less expressive uncertainty models than coherent sets of desirable gambles.



Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a sequence  $\omega_1, \dots, \omega_n$  of random variables to which  $\Omega$  is the finite set of possible values. So the possibility space is  $\Omega = \mathbb{R}^n$  and  $\omega = (\omega_1, \dots, \omega_n)$  is a list of  $n$  elements.

$\mathcal{D}_w$  is the set of all permutations  $\sigma$  of the index set  $\{1, \dots, n\}$ .  
The associated permutation of  $\mathbb{R}^n$  is defined by  $(\omega)_\sigma = \omega_{\sigma(i)}$ .  
It is called a permutation of  $\mathcal{D}(\mathbb{R}^n)$  by letting  $f \mapsto f \circ \sigma$ .

With every sequence of observations corresponds a joint vector in  $\mathbb{R}^n$ :  $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ .

The desirability  $f \in \mathcal{D}$  means  $f^*(\omega) \in \mathcal{D}$  implies  $f^*(\omega \circ \sigma) \in \mathcal{D}$  for a permutation  $\sigma$  of  $\mathbb{R}^n$ .  
Permitted sequences have the same joint vector: a permutation invariant gamble.



## Exchangeability

A subject assesses that  $\omega_1, \dots, \omega_n$  are exchangeable. This means that for any gamble  $f$  and any permutation  $\sigma$ , he finds exchanging  $f$  by  $f \circ \sigma$  fully desirable, because he is indifferent between them. The negative natural extension of all such exchange gambles is:  
 $\mathcal{D}_w = \{f \in \mathbb{R}^n \mid f \circ \sigma \in \mathcal{D}_w\}$ .

If  $\mathcal{D}_w$  consists of weakly desirable gambles, then so does its conical hull  $\text{conv}(\mathcal{D}_w)$  and  $\mathcal{D}_w = \text{conv}(\mathcal{D}_w)$ .

A subset  $\mathcal{A}$  of all exchange gambles on  $\mathbb{R}^n$  is called exchangeable if  $\text{conv}(\mathcal{A}) \subseteq \mathcal{D}_w$  or equivalently  $\mathcal{D}_w = \text{conv}(\mathcal{A})$ .

If  $\mathcal{A}$  is coherent and exchangeable then it is also permutation for  $\mathcal{A} \cap \mathcal{D}_w$  and all  $\sigma \in \mathcal{D}_w$  holds that  $f \circ \sigma \in \mathcal{A}$ .

## Exchangeable natural extension

The assessment of acyclic non-desirability under exchangeability if  $\mathcal{A} \cap \mathcal{D}_w$  avoids non-desirability.  
The exchangeable natural extension of  $\mathcal{A}$  is:  
 $\mathcal{D}(\mathcal{A}) := \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$ .

If  $\mathcal{A}$  is acyclic non-desirable under exchangeability, then  $\mathcal{D}(\mathcal{A})$  is the smallest non-desirable coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating exchangeable models

The subject observes the values  $\omega_1, \dots, \omega_n$  of the joint vector  $\omega$ . If  $\mathcal{A}$  is the list of variables  $\omega_1, \dots, \omega_n$ , the event observed is the event  $\{ \omega \mid \omega \in \mathcal{A} \} \subseteq \mathbb{R}^n$ . This event is observed in observations about the relationship  $\omega \in \mathcal{A}$  is desirable.

Contingent on observing  $\omega \in \mathcal{A}$ , the subject models his beliefs using updated sets of desirable gambles. The values of  $\mathcal{D}(B)$  are:  
 $\mathcal{D}(B) = \{g \in \mathbb{R}^n \mid g_{\mathcal{A}} \in \mathcal{D}_w\}$  and  
 $\mathcal{D}(B) = \{g \in \mathbb{R}^n \mid g_{\mathcal{A}} \in \mathcal{D} \text{ and } g_{\mathcal{A}^c} \in \mathcal{D}_w\}$ .

If  $\mathcal{A}$  is a coherent and exchangeable set of desirable gambles on  $\mathbb{R}^n$ , then  $\mathcal{D}(B)$  is a coherent and exchangeable set of desirable gambles on  $B$ .

Under exchangeability, joint vectors are sufficient statistics:  $\mathcal{D}(B) = \text{conv}(\mathcal{A} \cap \mathcal{D}_w)$ .

## Exchangeable previsions

A lower prevision  $\underline{E}(\mathbb{R}^n)$  is exchangeable if there is some non-desirable subset  $\mathcal{A}$  of desirable gambles  $\mathcal{A}$  such that  $\underline{E}(f) = \max\{c \in \mathbb{R} \mid f - c \cdot 1 \in \mathcal{D}(\mathcal{A})\}$ .

Representation

## Moving between sequence gambles and count gambles



The set of permutation invariant sequence gambles is  $\mathcal{D}_w(\mathbb{R}^n) = \{f \in \mathbb{R}^n \mid f \circ \sigma = f \text{ for all } \sigma \in \mathcal{D}_w\}$ .  
The projection of a sequence gamble  $f$  onto a permutation invariant sequence gamble is:  
 $\text{conv}(\mathcal{D}_w(\mathbb{R}^n)) \cap \text{conv}(\mathcal{D}_w(\mathbb{R}^n)) = \text{conv}(\mathcal{D}_w(\mathbb{R}^n))$ .

When its value on a realization  $\omega$  is given by:  
 $\text{conv}(\mathcal{D}_w(\mathbb{R}^n)) \cap \text{conv}(\mathcal{D}_w(\mathbb{R}^n)) = \text{conv}(\mathcal{D}_w(\mathbb{R}^n))$ .

The permutation invariant sequence gamble is a one-to-one correspondence with the count gamble  $f^*$ :  
 $f^*(\omega) = \sum_{i=1}^n \omega_i \cdot 1_i$ .

## Representation

A set of desirable gambles  $\mathcal{A}$  on  $\mathbb{R}^n$  is coherent and exchangeable if there is some coherent set  $\mathcal{D}$  of desirable gambles on  $\mathbb{R}^n$  is permutation-invariant such that:  
 $\mathcal{A} \subseteq \text{conv}(\mathcal{D}_w(\mathbb{R}^n))$ .

and in that case this  $\mathcal{D}$  is uniquely determined by:  
 $\mathcal{D} = \{f \in \mathbb{R}^n \mid f^*(\omega) \in \mathcal{D} \text{ for all } \omega \in \mathbb{R}^n\}$ .

## Exchangeable natural extension & representation

The assessment of acyclic non-desirability under exchangeability if  $\text{conv}(\mathcal{A}) \cap \mathcal{D}_w$  avoids non-desirability.

A count result  $\mathcal{A}^* = \{f^* \in \mathbb{R}^N \mid f^* \in \mathcal{A} \text{ for some } f \in \mathcal{A}\}$  is:  
 $\mathcal{A}^* = \{f^* \in \mathbb{R}^N \mid f^* \in \mathcal{A} \text{ for some } f \in \mathcal{A}\}$ .



## Representing updated models

The subject observes the values  $\omega_1, \dots, \omega_n$  of the joint vector  $\omega \in \mathbb{R}^n$  is  $\mathcal{A}$  if the list of variables  $\omega_1, \dots, \omega_n$ .

If  $\mathcal{A}$  is a coherent and exchangeable set of desirable gambles on  $\mathbb{R}^n$ , then the representation of the lower prevision of exchangeable updated models is given by:  
 $\underline{E}(f) = \max\{c \in \mathbb{R} \mid f - c \cdot 1 \in \mathcal{D}(\mathcal{A})\}$ .

This representation is not an updated model of the representation  $\underline{E} = \text{conv}(\mathcal{D}_w(\mathbb{R}^n))$  of  $\mathcal{A}$ . They are however related by:  
 $\underline{E}(f) = \max\{c \in \mathbb{R} \mid f - c \cdot 1 \in \mathcal{D}(\mathcal{A})\}$ .

When we use the identical function, defined for every count vector  $\omega \in \mathbb{R}^n$  by:  
 $\omega \mapsto \sum_{i=1}^n \omega_i \cdot 1_i$ .

which is also when  $\omega \in \mathcal{A}$ .

## Exchangeable previsions & representation

A lower prevision  $\underline{E}$  on  $\mathbb{R}^n$  is coherent and exchangeable if there is some coherent lower prevision  $\underline{E}^*$  on  $\mathbb{R}^N$  such that:  
 $\underline{E}(f) = \max\{c \in \mathbb{R} \mid f - c \cdot 1 \in \mathcal{D}(\mathcal{A})\}$ .

in that case  $\underline{E}^*$  is uniquely determined by  $\underline{E}^*(f^*) = \underline{E}(f)$ .

Gert de Cooman & Erik Quaeghebeur

SYSTeMS Research Group & FUM Research Unit, Ghent University

{Gert.deCooman, Erik.Quaeghebeur}@UGent.be



- ▶ representation theorem
- ▶ exchangeable natural extension & representation
- ▶ representing updated exchangeable models
- ▶ relationship with representing previsions

# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\Omega$  of outcomes of an experiment.  
A subject sets a valuation about the experiment's outcome.  
Standard  $f: \Omega \rightarrow \mathbb{R}^n$  interpreted as gambles means:  
 $f(\omega)$  is the experiment's outcome is  $\omega$ .



A gamble  $f$  is desirable to the subject if he accepts the following bet: observe  $\omega$  in  $\Omega_1$  if the actual outcome  $\omega$  is determined, and if the subject's gamble is changed by  $f(\omega)$ .  
The new gamble  $g$  is not desirable.

## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{D}$  is desirable if it meets the experiment's outcome.



The set of desirable gambles  $\mathcal{D}$  is coherent if it satisfies the following stability requirements:  
D1.  $0 \in \mathcal{D}$   
D2.  $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$   
D3.  $f \in \mathcal{D} \Rightarrow \lambda f \in \mathcal{D}$  for  $\lambda > 0$   
D4.  $f \in \mathcal{D} \Rightarrow f \vee g \in \mathcal{D}$  for any gamble  $g$   
D5.  $f \in \mathcal{D} \Rightarrow f \wedge g \in \mathcal{D}$  for any gamble  $g$   
D6.  $f, g \in \mathcal{D} \Rightarrow f \vee g \in \mathcal{D}$  for any gamble  $g$   
Requirements D2 and D5 make  $\mathcal{D}$  a convex cone in  $\mathbb{R}^n$ .

## Sets of weakly desirable gambles

The subject considers a gamble  $f$  in  $\mathcal{D}$  weakly desirable if he would accept gamble  $f$  & another desirable gamble  $g$  observed,  $g \vee f \in \mathcal{D}$  then  $f \vee g \in \mathcal{D}$ .

The subject's set of weakly desirable gambles is  $\mathcal{D}_w = \{f \in \mathcal{D} \mid f \vee g \in \mathcal{D}\}$ .

The set of weakly desirable gambles  $\mathcal{D}_w$  corresponding to a coherent set of desirable gambles  $\mathcal{D}$  satisfies the following properties:  
W1.  $0 \in \mathcal{D}_w$   
W2.  $f, g \in \mathcal{D}_w \Rightarrow f + g \in \mathcal{D}_w$  (additive property)  
W3.  $f \in \mathcal{D}_w \Rightarrow \lambda f \in \mathcal{D}_w$  for  $\lambda > 0$  (positive homogeneity)  
W4.  $f, g \in \mathcal{D}_w \Rightarrow f \vee g \in \mathcal{D}_w$  (maximal property)  
W5.  $f, g \in \mathcal{D}_w \Rightarrow f \wedge g \in \mathcal{D}_w$  (minimal property)

$\mathcal{D}_w$  is the closure of  $\mathcal{D}$  including gambles in  $\mathcal{K}_c(0)$ .

## Assessments & their natural extension

An assessment can consist of a set  $\mathcal{A} \subseteq \mathcal{D}_w(0)$  considered desirable by the subject.

The assessment of a gamble  $f$  is non-positive if the intersection of  $\text{conv}(\mathcal{A})$  and  $\mathcal{K}_c(0)$  is empty.

The natural extension  $\mathcal{D}^*$  of  $\mathcal{A}$  is

$$\mathcal{D}^*(f) := \text{conv}(\mathcal{A} \cup \{f\}) \cap \mathcal{K}_c(0)$$

If  $\mathcal{A}$  is desirably non-positive, then  $\mathcal{D}^*(f)$  is the smallest coherent set of desirable gambles including  $f$ .

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \Omega$ .

Contingent on observing  $B$ , the subject updates his beliefs using an updated set of desirable gambles. The subject's  $\mathcal{D}^*(B)$  given by

$$\mathcal{D}^*(B) := \{f \in \mathcal{D} \mid f \wedge \mathbb{1}_B \in \mathcal{D}\}$$

If  $\mathcal{D}$  is a coherent set of desirable gambles on  $\Omega$ , then  $\mathcal{D}^*(B)$  is a coherent set of desirable gambles on  $B$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  is associated to a set of desirable gambles  $\mathcal{D}$  by

$$\underline{P}(f) := \max\{p \in \mathbb{R} \mid f - p \in \mathcal{D}\}$$

Its conjugate upper prevision  $\overline{P}(f)$  is defined by  $\overline{P}(f) := -\underline{P}(-f)$ .

A lower prevision  $\underline{P}$  is coherent if there exists some coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{P}(f) = \max\{p \in \mathbb{R} \mid f - p \in \mathcal{D}\}$ .

Coherent lower previsions are best representable uncertainty models, their coherent sets of desirable gambles.

Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a random  $X_1, \dots, X_n$  of random variables to which  $\mathcal{D}$  is the finite set of possible bets. So the possibility  $\mathcal{D}$  is  $\mathcal{D}^* = \text{conv}\{x_1, \dots, x_n\}$  for  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ .

$\mathcal{D}_w$  is the set of all permutations of the finite set  $\{x_1, \dots, x_n\}$ .

The associated permutation  $\mathcal{D}^*$  is defined by  $\mathcal{D}^*(x) = \text{conv}\{x \circ \pi \mid \pi \in \text{Perm}(n)\}$  if  $x$  is a permutation of  $\mathcal{D}$  or  $\mathcal{D}^*(x) = \emptyset$  otherwise.

With every sequence of observations corresponds a joint vector in  $\mathbb{R}^n$   $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ .

The desirability  $\underline{P}(x)$  is  $\underline{P}(x) = \max\{p \in \mathbb{R} \mid x - p \in \mathcal{D}\}$ .

Permitted gambles have the same joint vector: a permutation element  $g$  is

$$g = (x_{\pi(1)}, \dots, x_{\pi(n)}) \in \mathcal{D}^*$$

## Moving between sequence gambles and count gambles



The set of permutation invariant sequence gambles is  $\mathcal{D}_w = \{f \in \mathcal{D} \mid f \circ \pi = f \text{ for all } \pi \in \text{Perm}(n)\}$ .

The projection of a sequence gamble  $f$  onto a permutation invariant sequence gamble is

$$\text{proj}(f) := \sum_{\pi \in \text{Perm}(n)} \text{Mult}(\pi^{-1}(f)) \circ \pi$$

whose value on an individual  $\omega$  is given by

$$\text{proj}(f)(\omega) := \sum_{\pi \in \text{Perm}(n)} f(\omega \circ \pi)$$

The count gamble corresponding to the sequence gamble  $f$  is  $\text{Mult}(f)(\omega) := \text{Mult}(f)(\omega)$ .

The permutation invariant sequence gamble is a one-to-one correspondence with the count gamble via  $\text{Mult}(g) = \text{proj}(f)$ .

## Exchangeability

A subject assesses that  $X_1, \dots, X_n$  are exchangeable. This means that for any gamble  $f$  and any permutation  $\pi$  he finds exchanging  $f$  by  $f \circ \pi$  weakly desirable, because he is indifferent between them. The negative mean response of all such exchange gambles is

$$\mathcal{D}_w = \{f \in \mathcal{D} \mid f \circ \pi \in \mathcal{D}\}$$

If  $\mathcal{D}_w$  consists of weakly desirable gambles, then so does its conical hull  $\mathcal{D}_w^* = \text{conv}(\mathcal{D}_w)$ .

A coherent set of sequence gambles on  $\mathcal{D}^*$  is called exchangeable  $\mathcal{D}^* \subseteq \mathcal{D}_w^*$  or equivalently  $\mathcal{D}_w^* \subseteq \mathcal{D}^*$ .

$\mathcal{D}^*$  is coherent and exchangeable if and only if permutation for  $\mathcal{D}^*$  is  $\mathcal{D}^*$  and  $\mathcal{D}^*$  is  $\mathcal{D}_w^*$ . It holds that  $\mathcal{D}^* \subseteq \mathcal{D}_w^*$ .

## Exchangeable natural extension

The assessment of a gamble  $f$  is non-positive under exchangeability if  $f \in \mathcal{D}_w^*$  avoids non-positivity.

The exchangeable natural extension of  $f$  is

$$\mathcal{D}^*(f) := \text{conv}(\mathcal{D}_w^* \cup \{f\}) \cap \mathcal{K}_c(0)$$

If  $\mathcal{A}$  is desirably non-positive under exchangeability, then  $\mathcal{D}^*(f)$  is the smallest exchangeable coherent set of desirable gambles including  $f$ .

## Updating exchangeable models

The subject observes the values  $x = (x_1, \dots, x_n)$  of the joint vector  $x$ . If  $\mathcal{D}^*$  is the best  $\mathcal{D}_w^*$  available,  $\mathcal{K}_c(x)$  is the mean observing the event  $\{x \mid \mathcal{D}^* \cap \mathcal{K}_c(x) \neq \emptyset\}$ . This set is intended to be relevant about the relationship  $x \in \mathcal{D}^*$ .

Contingent on observing  $x$  or  $\mathcal{K}_c(x)$ , the subject updates his beliefs using updated sets of desirable gambles. The subject's  $\mathcal{D}^*(x)$  that are

$$\mathcal{D}^*(x) := \{f \in \mathcal{D}_w^* \mid f \wedge \mathbb{1}_x \in \mathcal{D}_w^*\}$$

If  $\mathcal{D}^*$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{D}^*$ , then  $\mathcal{D}^*(x)$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{K}_c(x)$ .

Under exchangeability, count vectors are sufficient statistics:  $\mathcal{D}^*(x) = \text{conv}(\mathcal{D}_w^* \cap \mathcal{K}_c(x))$ .

## Exchangeable previsions

A lower prevision  $\underline{P}$  on  $\mathcal{D}^*$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{D}^*$  such that  $\underline{P}(f) = \max\{p \in \mathbb{R} \mid f - p \in \mathcal{D}^*\}$ .

Representation

## Representation

A set of desirable gambles  $\mathcal{D}$  on  $\mathcal{D}^*$  is coherent and exchangeable if there is some coherent set  $\mathcal{D}^*$  of desirable gambles on  $\mathcal{D}^*$  - its dual representation - such that

$$\mathcal{D} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

and in that case this  $\mathcal{D}^*$  is uniquely determined by

$$\mathcal{D}^* = \{g \in \mathcal{D}_w^* \mid \text{proj}(g) \in \mathcal{D}\} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

## Exchangeable natural extension & representation

The assessment of a gamble  $f$  is non-positive under exchangeability if  $\text{Mult}(f)(\omega) \in \mathcal{D}_w^*$  avoids non-positivity.

A coherent  $\text{Mult}(\mathcal{D}_w^*) \circ \text{proj} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$ .

$$\text{Mult}(\mathcal{D}_w^*) \circ \text{proj} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

$$\text{Mult}(\mathcal{D}_w^*) \circ \text{proj} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

$$\text{Mult}(\mathcal{D}_w^*) \circ \text{proj} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

$$\text{Mult}(\mathcal{D}_w^*) \circ \text{proj} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

$$\text{Mult}(\mathcal{D}_w^*) \circ \text{proj} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

$$\text{Mult}(\mathcal{D}_w^*) \circ \text{proj} = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

## Representing updated models

The subject observes the values  $x = (x_1, \dots, x_n)$  of the joint vector  $x$ . If  $\mathcal{D}^*$  is the best  $\mathcal{D}_w^*$  available,  $\mathcal{K}_c(x)$  is the mean observing the event  $\{x \mid \mathcal{D}^* \cap \mathcal{K}_c(x) \neq \emptyset\}$ .

If  $\mathcal{D}^*$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{D}^*$ , then the representation of the lower - because of exchangeability - identical updated models is given by

$$\mathcal{D}^*(x) = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

This representation is not an updated model of the representation  $\mathcal{D}^* = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$  of  $\mathcal{D}^*$ . They are however related by

$$\mathcal{D}^*(x) = \text{Mult}(\mathcal{D}^*) \circ \text{proj}$$

when we use the identical function, defined for every count vector  $x$  by  $\text{proj}(x) := \sum_{\pi \in \text{Perm}(n)} \text{Mult}(\pi^{-1}(x)) \circ \pi$ .

which is also true if  $x$  is

$$\text{proj}(x) := \sum_{\pi \in \text{Perm}(n)} \text{Mult}(\pi^{-1}(x)) \circ \pi$$

Exchangeable previsions & representation

A lower prevision  $\underline{P}$  on  $\mathcal{D}^*$  is coherent and exchangeable if there is some coherent lower prevision  $\underline{P}^*$  on  $\mathcal{D}^*$  - its exchangeable representation - such that  $\underline{P}(f) = \max\{p \in \mathbb{R} \mid f - p \in \mathcal{D}^*\}$ . In that case  $\underline{P}^*$  is uniquely determined by  $\underline{P}^*(g) = \underline{P}(\text{proj}(g))$ .

Best definition of coherence?

Infinite exchangeable sequences?

Representation for infinite exchangeable sequences?

# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\mathcal{X}$  of outcomes of some experiment.  
A subject sets a collection about the experiment's outcome.  
Standard  $f: \mathcal{X} \rightarrow \mathbb{R}^n$  interpreted as gambler's reward:  
 $f(x)$  when the experiment's outcome is  $x$ .



A gambler  $\mathcal{G}$  decides to bet subject  $f$  if she satisfies the following desiderata:  
(1) The actual outcome  $x$  is determined, and  
(2) The subject's capital is changed by  $f(x)$ .  
The new gambler  $\mathcal{G}'$  is not desirable.

## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{G} \subseteq \mathcal{F}(\mathcal{X})$  models his beliefs about the experiment's outcome.

- The set of desirable gambles  $\mathcal{G}$  is coherent if it satisfies the following desiderata requirements:
- $\mathcal{G}$  is a convex cone.
  - $\mathcal{G}$  is closed under addition of non-negative multiples of its elements.
  - $\mathcal{G}$  is closed under multiplication by non-negative scalars.
  - $\mathcal{G}$  is closed under multiplication by non-negative scalars.
  - $\mathcal{G}$  is closed under multiplication by non-negative scalars.
  - $\mathcal{G}$  is closed under multiplication by non-negative scalars.



## Sets of weakly desirable gambles

The subject considers a gamble  $f$  to be  $\mathcal{W}$ -desirable if by adding any desirable gamble  $g$  to  $f$ , another desirable gamble is obtained, i.e.  $f + g \in \mathcal{W}$  when  $f \in \mathcal{W}$  and  $g \in \mathcal{G}$ .

The subject's set of weakly desirable gambles is  $\mathcal{W} = \{f \in \mathcal{F}(\mathcal{X}) \mid f + g \in \mathcal{W}, \forall g \in \mathcal{G}\}$ .  
The set of weakly desirable gambles  $\mathcal{W}$ , corresponding to a coherent set of desirable gambles  $\mathcal{G}$ , satisfies the following properties:  
(1)  $\mathcal{W}$  is a convex cone.  
(2)  $\mathcal{W}$  is closed under addition of non-negative multiples of its elements.  
(3)  $\mathcal{W}$  is closed under multiplication by non-negative scalars.  
(4)  $\mathcal{W}$  is closed under multiplication by non-negative scalars.  
(5)  $\mathcal{W}$  is closed under multiplication by non-negative scalars.



## Assessments & their natural extension

An assessment can consist of a set  $\mathcal{A} \subseteq \mathcal{F}(\mathcal{X})$  considered desirable by the subject.

The assessment of a gamble  $f$  is non-positive if the intersection of  $\text{conv}(\mathcal{A})$  and  $\mathcal{K}_c(\mathcal{A})$  is empty. The natural extension of  $\mathcal{A}$  is  $\mathcal{E}(\mathcal{A}) = \text{conv}(\mathcal{A}) \cap \mathcal{K}_c(\mathcal{A})$ .

If  $\mathcal{A}$  is desicc non-positive, then  $\mathcal{E}(\mathcal{A})$  is the smallest coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \mathcal{X}$ .  
Contingent on observing  $B$ , the subject models his beliefs using a coherent set of desirable gambles. The subject's  $\mathcal{W}(B)$  given by  $\mathcal{W}(B) = \{f \in \mathcal{F}(\mathcal{X}) \mid f + g \in \mathcal{W}, \forall g \in \mathcal{G}\}$ .

If  $\mathcal{W}$  is a coherent set of desirable gambles on  $\mathcal{X}$ , then  $\mathcal{W}(B)$  is a coherent set of desirable gambles on  $\mathcal{X}$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  established by a set of desirable gambles  $\mathcal{G}$  is  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

Its conjugate upper prevision  $\overline{P}(f)$  is  $\overline{P}(f) = \sup\{x \in \mathbb{R} \mid x - f \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  is coherent if there exists a coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

Coherent lower previsions are best representable uncertainty models, their coherent sets of desirable gambles.



Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a random  $X_1, \dots, X_n$  of random variables to which  $\mathcal{X}$  is the finite set of possible values. So the possibility space is  $\mathcal{X} = \{(\omega_1, \dots, \omega_n) \mid \omega_i \in \mathcal{X}_i, \forall i=1, \dots, n\}$ .

$\mathcal{F}_n$  is the set of all permutations  $\pi$  of the index set  $\{1, \dots, n\}$ .  
The associated permutation of  $\mathcal{X}$  is defined by  $(\omega_1, \dots, \omega_n) \mapsto (\omega_{\pi(1)}, \dots, \omega_{\pi(n)})$ .

It is called a permutation of  $\mathcal{X}$  if  $\pi \in \mathcal{F}_n$  by letting  $f \mapsto f \circ \pi$ .

With every sequence of observations corresponds a joint vector in  $\mathbb{R}^n$ :  $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ .

The corresponding  $\mathcal{F}_n$  is  $\mathcal{F}_n = \{\pi \in \mathcal{F}_n \mid \pi(1) = 1, \dots, \pi(n) = n\}$ .

Permitted gambles have the same joint vector: a permutation invariant gamble.

$$\mathcal{G} = \{g \in \mathcal{F}_n \mid g(\omega) = g(\omega \circ \pi), \forall \omega \in \mathcal{X}, \forall \pi \in \mathcal{F}_n\}$$

## Exchangeability

A subject assesses that  $X_1, \dots, X_n$  are exchangeable. This means that for any gamble  $g$  and any permutation  $\pi$  of  $\mathcal{X}$  for  $g$  weakly desirable, because for  $\pi$  indifferent between them. The negative mean-representation of each such exchange gamble is  $\mathcal{W} = \{f \in \mathcal{F}_n \mid f + g \in \mathcal{W}, \forall g \in \mathcal{G}\}$ .

$\mathcal{W}$  consists of weakly desirable gambles, then so does its conical hull  $\mathcal{W}_c = \text{conv}(\mathcal{W}) \cap \mathcal{K}_c(\mathcal{W})$ .

A coherent set of desirable gambles on  $\mathcal{X}^n$  is called exchangeable if  $\mathcal{W}_c \subseteq \mathcal{W}$  or equivalently  $\mathcal{W}_c = \mathcal{W}$ .

If  $\mathcal{G}$  is coherent and exchangeable then it is also permutation for all  $\pi \in \mathcal{F}_n$  and all  $f \in \mathcal{G}$ , holds that  $f \circ \pi \in \mathcal{G}$ .

## Exchangeable natural extension

The assessment of a gamble  $f$  is non-positive under exchangeability if  $f \in \mathcal{W}_c$  avoids non-positivity.

The exchangeable natural extension of  $\mathcal{A}$  is  $\mathcal{E}(\mathcal{A}) = \text{conv}(\mathcal{A}) \cap \mathcal{K}_c(\mathcal{A})$ .

If  $\mathcal{A}$  is desicc non-positive under exchangeability, then  $\mathcal{E}(\mathcal{A})$  is the smallest exchangeable coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating exchangeable models

The subject observes the values  $x = (x_1, \dots, x_n)$  on the joint vector  $\omega$ , i.e.  $\omega$  is the first  $n$  coordinates  $X_1, \dots, X_n$  of the mean observing the event  $\{(\omega_1, \dots, \omega_n) \mid \omega_i = x_i, \forall i=1, \dots, n\}$ . We are interested in observations about the relationship  $x \in \mathcal{B}$  or not.

Contingent on observing  $x$  or  $\mathcal{B}$ , the subject models his beliefs using updated sets of desirable gambles. The subject's  $\mathcal{W}(x)$  that are  $\mathcal{W}(x) = \{f \in \mathcal{F}_n \mid f + g \in \mathcal{W}, \forall g \in \mathcal{G}\}$ .

Contingent on observing  $x$  or  $\mathcal{B}$ , the subject models his beliefs using updated sets of desirable gambles. The subject's  $\mathcal{W}(x)$  that are  $\mathcal{W}(x) = \{f \in \mathcal{F}_n \mid f + g \in \mathcal{W}, \forall g \in \mathcal{G}\}$ .

If  $\mathcal{W}$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{X}^n$ , then  $\mathcal{W}(x)$  and  $\mathcal{W}(B)$  are coherent and exchangeable sets of desirable gambles on  $\mathcal{X}^n$ .

Under exchangeability, joint vectors are sufficient statistics:  $\mathcal{W}(x) = \text{conv}(\mathcal{W}(x)) \cap \mathcal{K}_c(\mathcal{W}(x))$ .

## Exchangeable previsions

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

## Moving between sequence gambles and count gambles



The set of permutation invariant sequence gambles is  $\mathcal{G}_n = \{g \in \mathcal{F}_n \mid g(\omega) = g(\omega \circ \pi), \forall \omega \in \mathcal{X}, \forall \pi \in \mathcal{F}_n\}$ .

The projection of a sequence gamble  $f$  onto a permutation invariant sequence gamble is  $\mathcal{P}(f) = \sum_{\pi \in \mathcal{F}_n} f \circ \pi / |\mathcal{F}_n|$ .

The count gamble corresponding to the sequence gamble  $f$  is  $\mathcal{C}(f) = \sum_{\omega \in \mathcal{X}} f(\omega) \mathbb{1}_{\omega}$ .

The permutation invariant sequence gamble  $f$  is one-to-one correspondence with the count gamble  $\mathcal{C}(f)$ .

$$\mathcal{P}(\mathcal{C}(f)) = \mathcal{C}(\mathcal{P}(f))$$

## Representation

A set of desirable gambles  $\mathcal{G}$  on  $\mathcal{X}^n$  is coherent and exchangeable if there is some coherent set  $\mathcal{A}$  of desirable gambles on  $\mathcal{X}^n$  - its desicc representation - such that  $\mathcal{G} = \text{conv}(\mathcal{A}) \cap \mathcal{K}_c(\mathcal{A})$ .

and in that case this  $\mathcal{A}$  is uniquely determined by  $\mathcal{A} = \{f \in \mathcal{F}_n \mid f + g \in \mathcal{G}, \forall g \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

## Exchangeable natural extension & representation

The assessment of a gamble  $f$  is desicc non-positive under exchangeability if  $f \in \mathcal{W}_c$  avoids non-positivity.

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

## Representing updated models

The subject observes the values  $x = (x_1, \dots, x_n)$  on the joint vector  $\omega$ , i.e.  $\omega$  is the first  $n$  coordinates  $X_1, \dots, X_n$  of the mean observing the event  $\{(\omega_1, \dots, \omega_n) \mid \omega_i = x_i, \forall i=1, \dots, n\}$ .

If  $\mathcal{W}$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{X}^n$ , then  $\mathcal{W}(x)$  and  $\mathcal{W}(B)$  are coherent and exchangeable sets of desirable gambles on  $\mathcal{X}^n$ .

Under exchangeability, joint vectors are sufficient statistics:  $\mathcal{W}(x) = \text{conv}(\mathcal{W}(x)) \cap \mathcal{K}_c(\mathcal{W}(x))$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

A lower prevision  $\underline{P}$  on  $\mathcal{F}_n$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{G}$  such that  $\underline{P}(f) = \inf\{x \in \mathbb{R} \mid f - x \in \mathcal{G}\}$ .

See you at the poster!