# Dutch Books and Combinatorial Games

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# About myself

Bachelor in archaeology and Master in Mathematics from University of Copenhagen.

Thesis on minimizing information divergence. Worked as archaeologist.

PhD in natural sciences from University of Roskilde with a thesis entitled "Time and Conditional Independence".

For 5 year full time mountaineer. Wrote 7 books. Member of the safety commision of the International Climbing and Mountaineering Federation. President of Danish Climbing Federation.

6 years at University of Copenhagen and research fellow in Bielefeld.
Now researcher at CWI and editor-in-chief of the journal Entropy.

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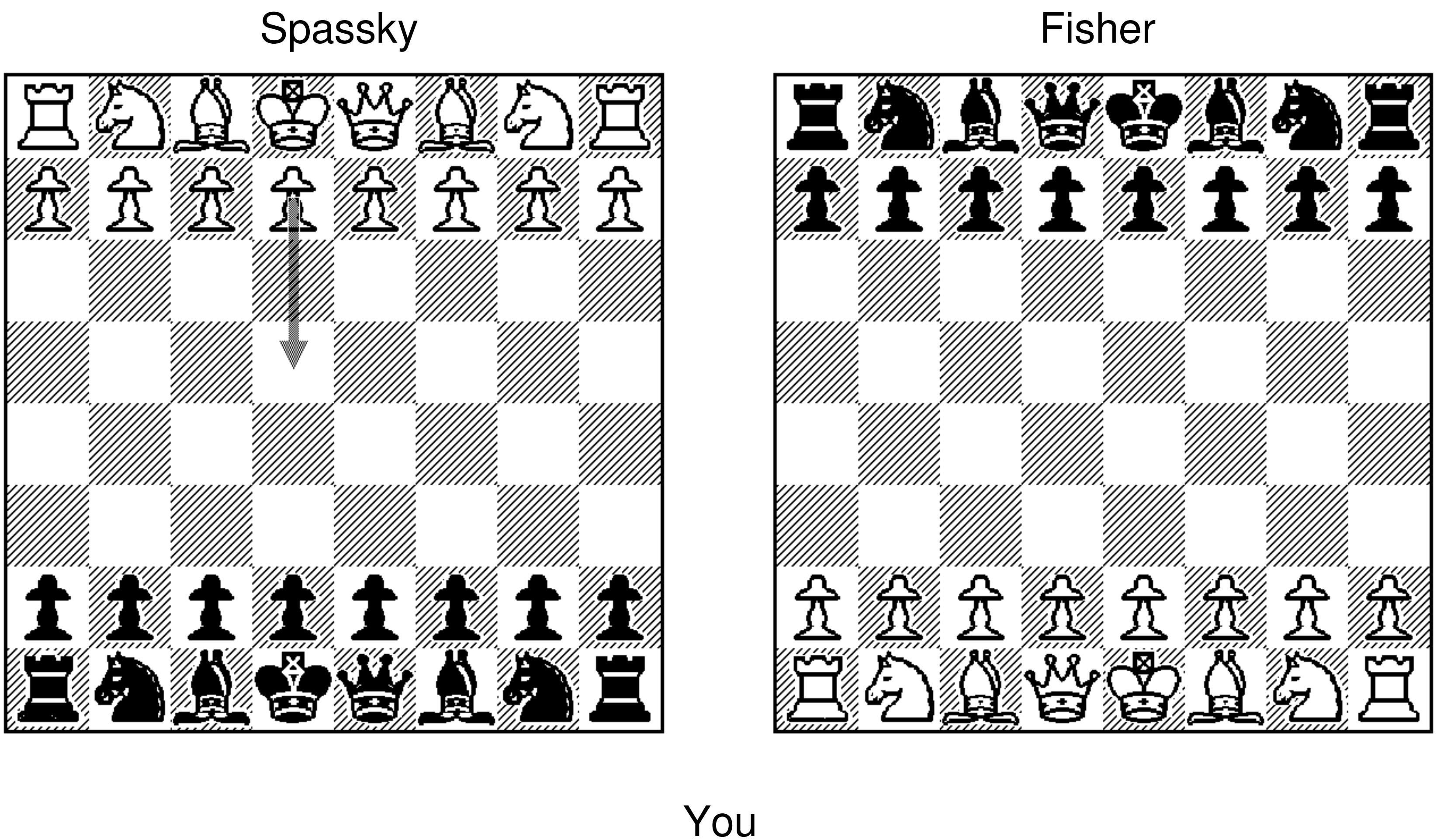
# How to win over a chess grand master



# How to win over a chess grand master

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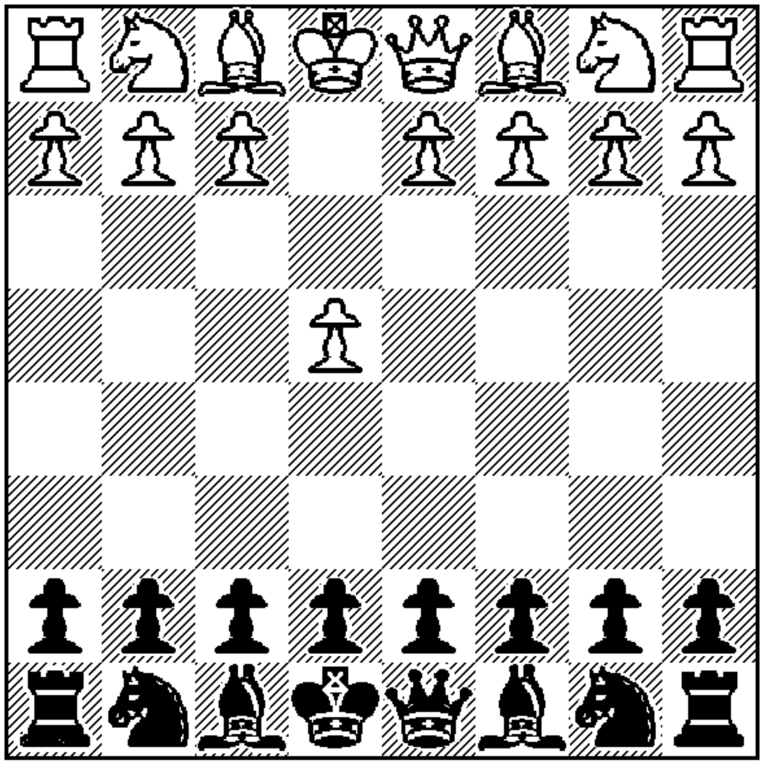

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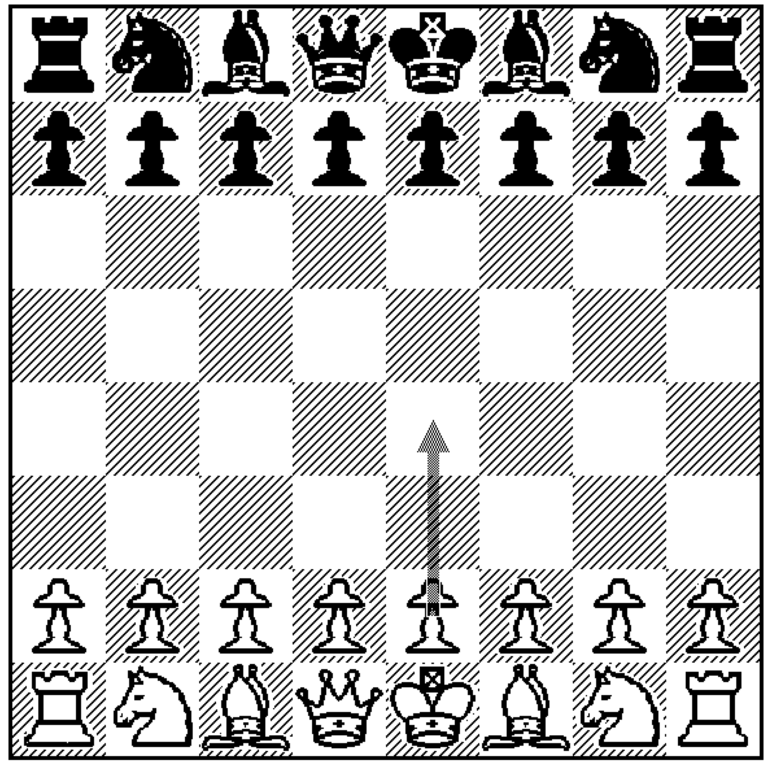

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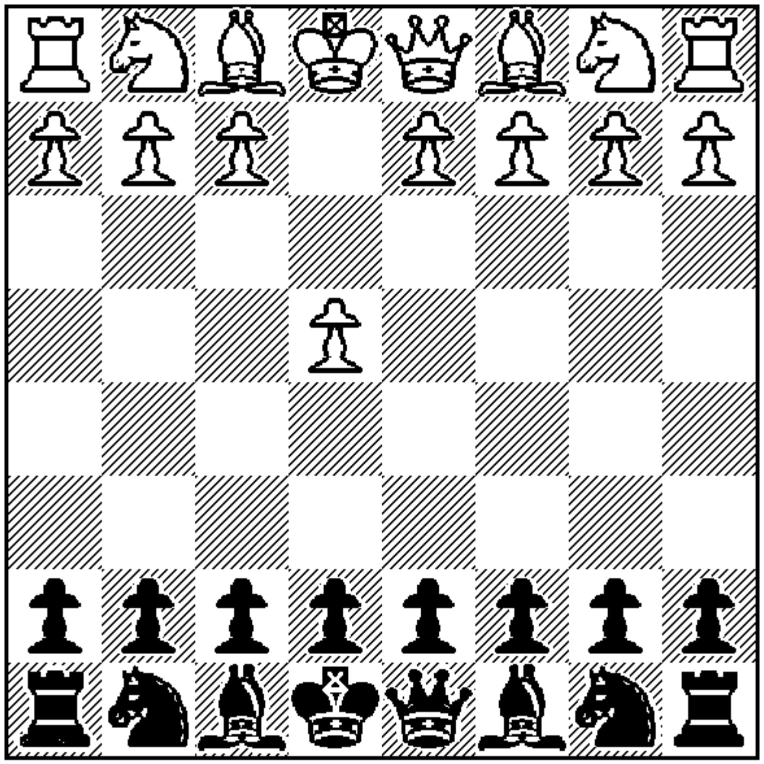

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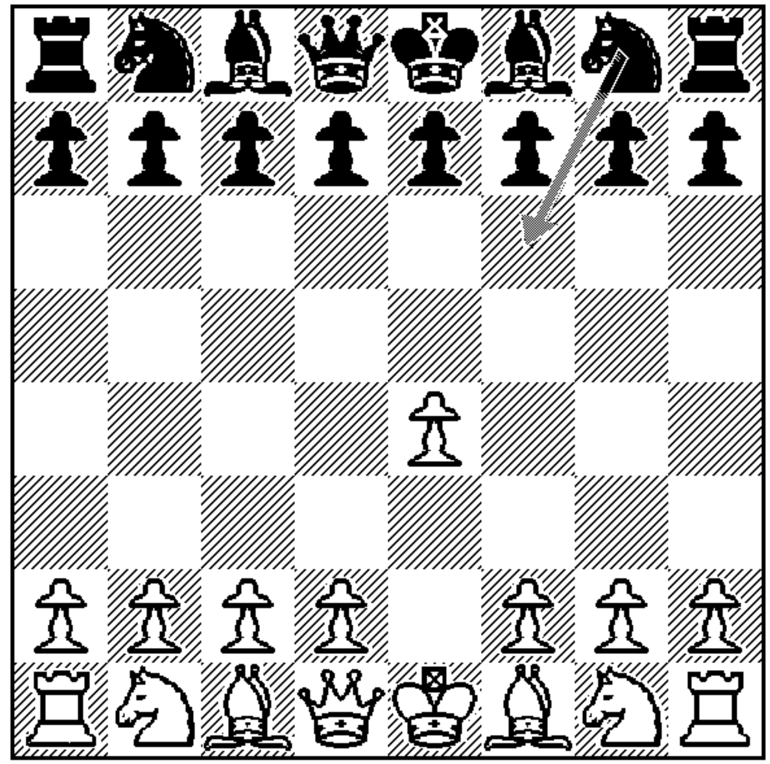

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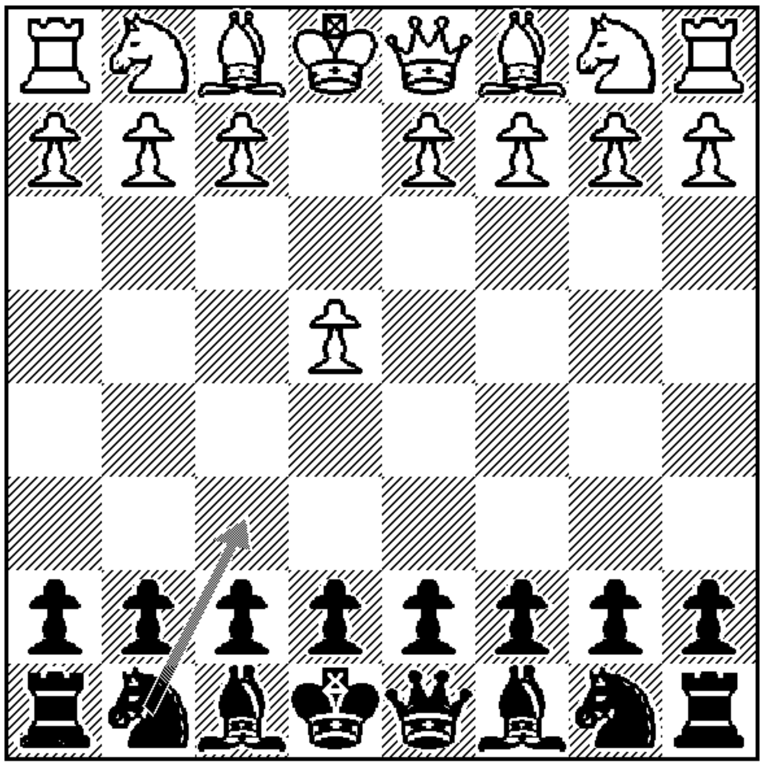

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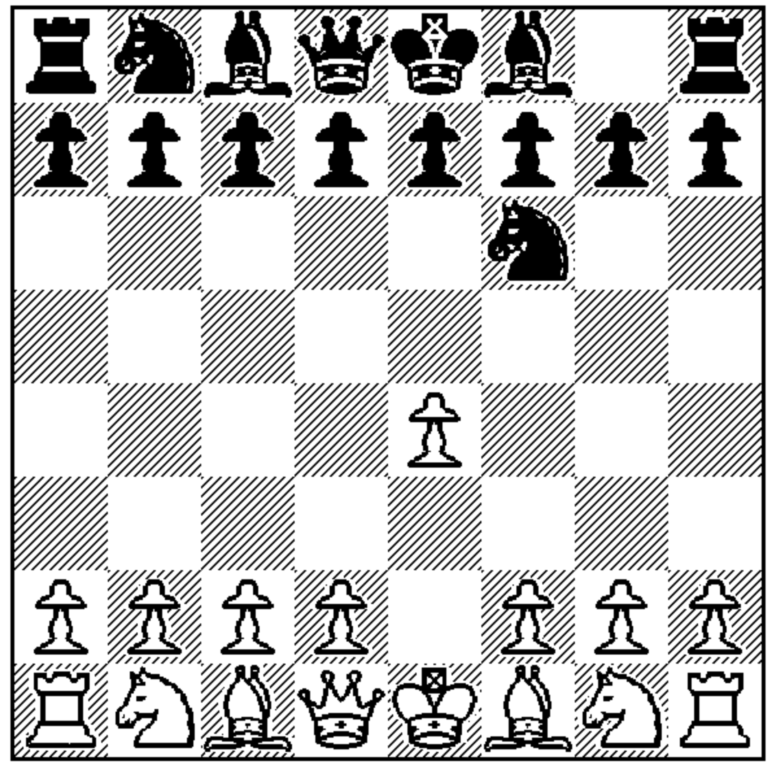

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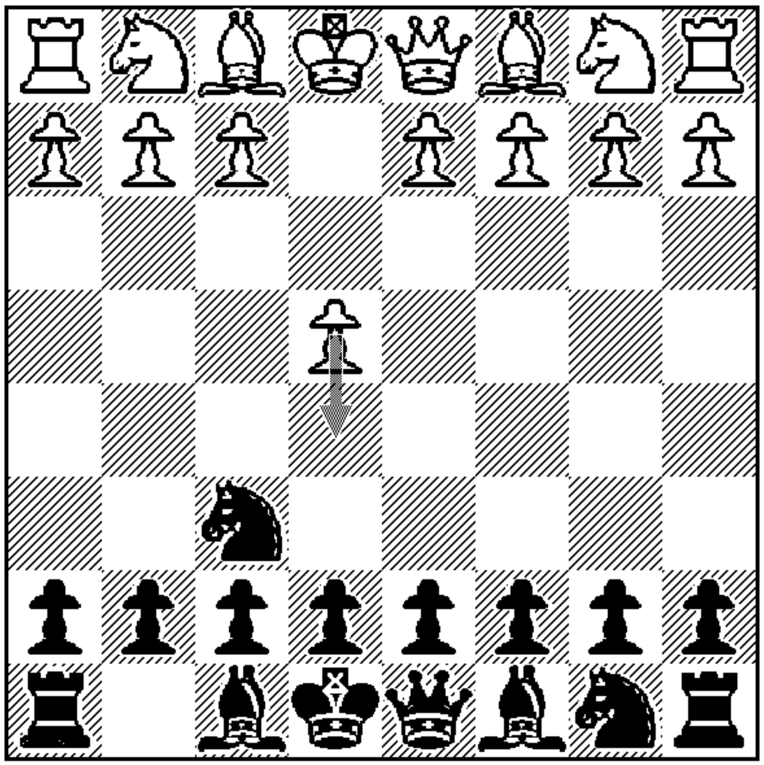


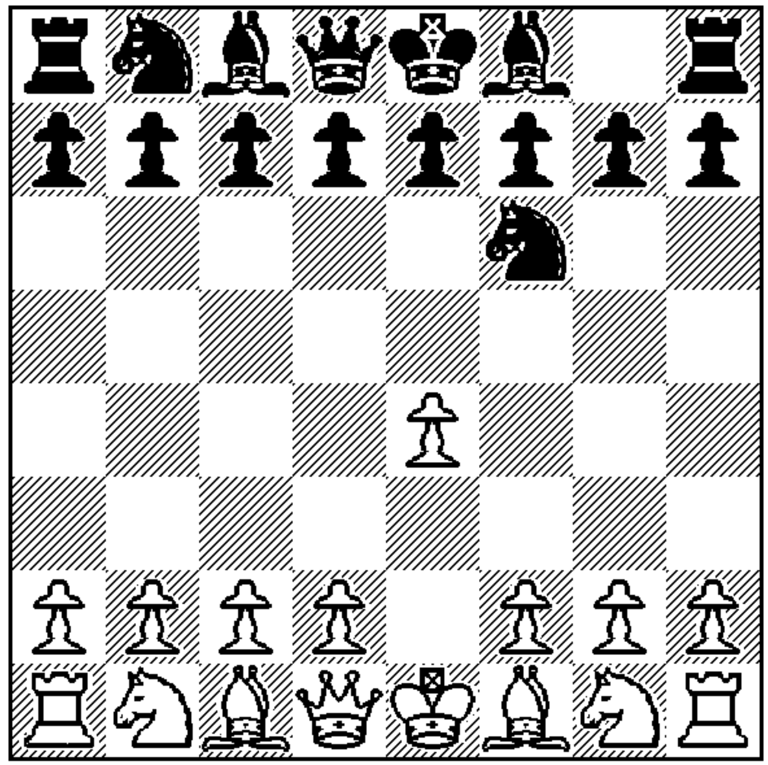


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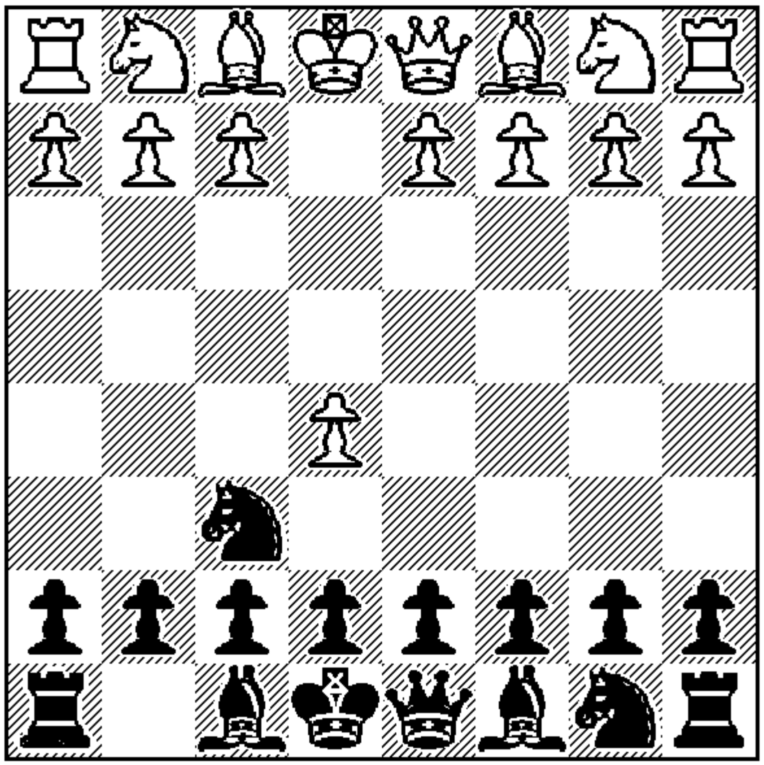

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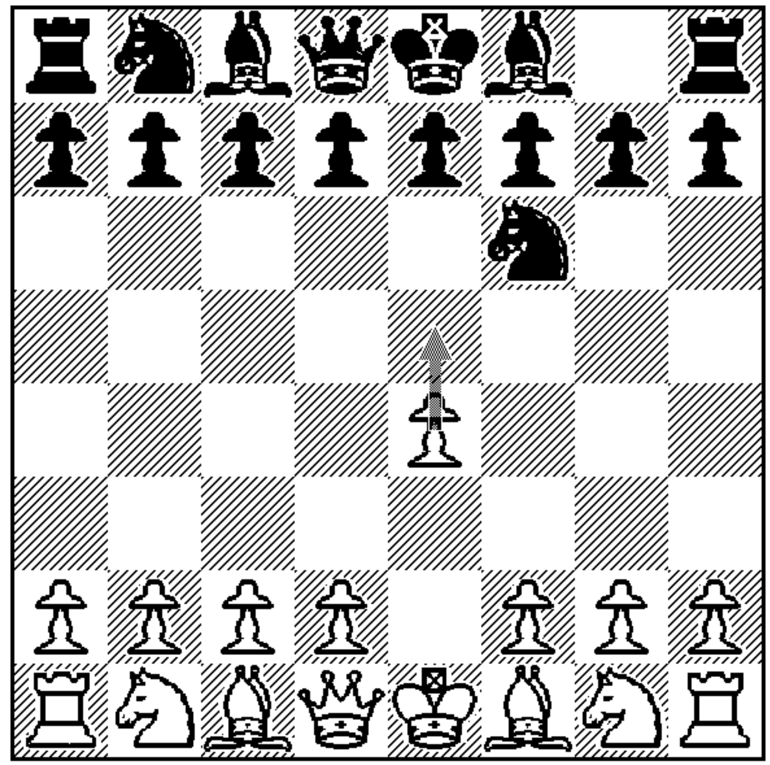

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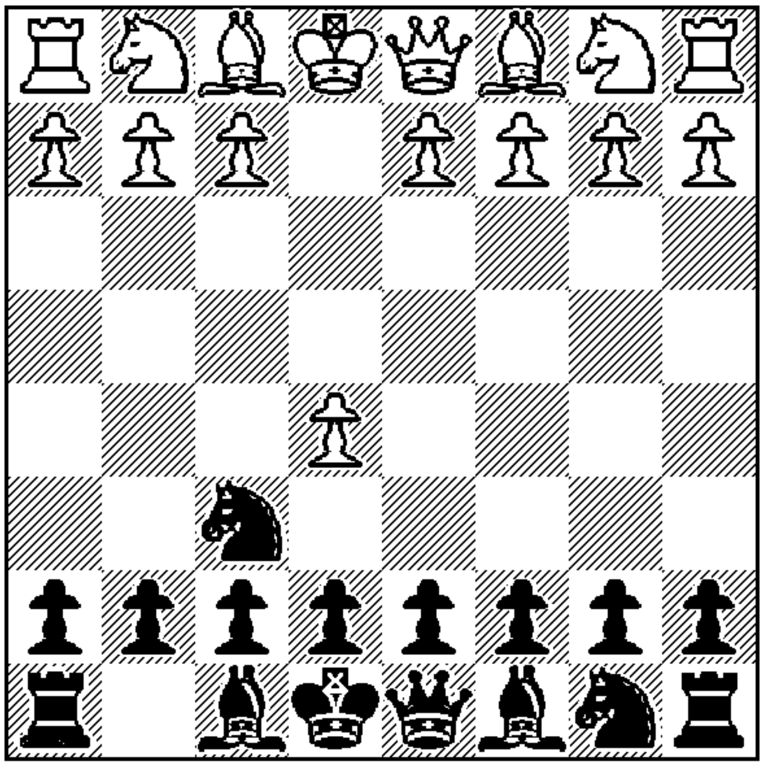


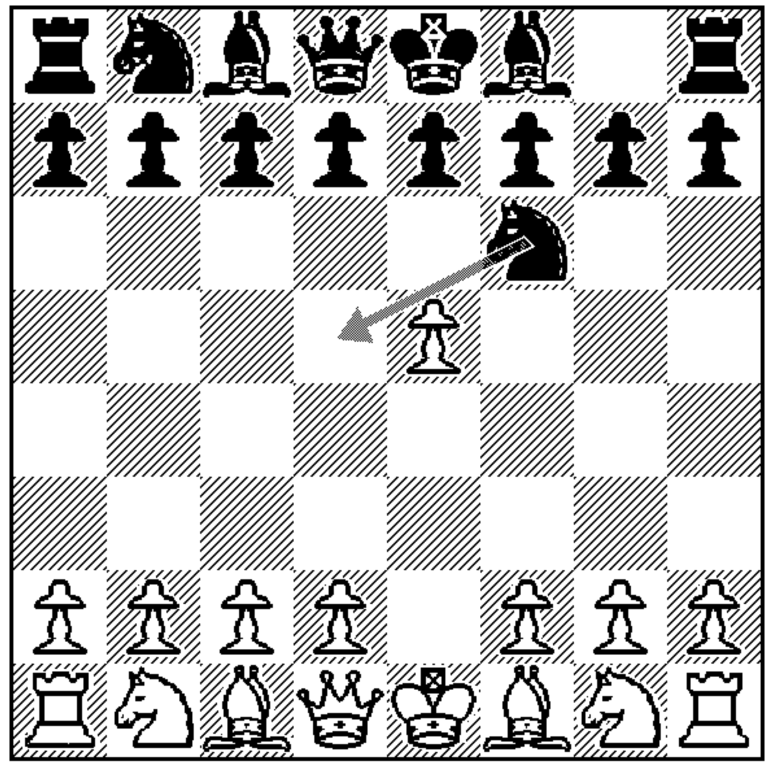


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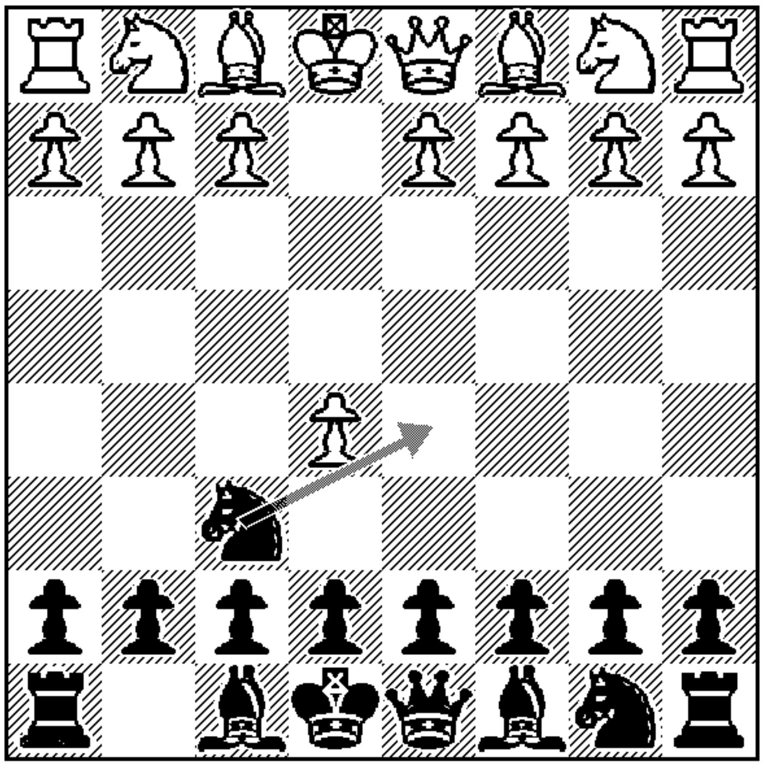

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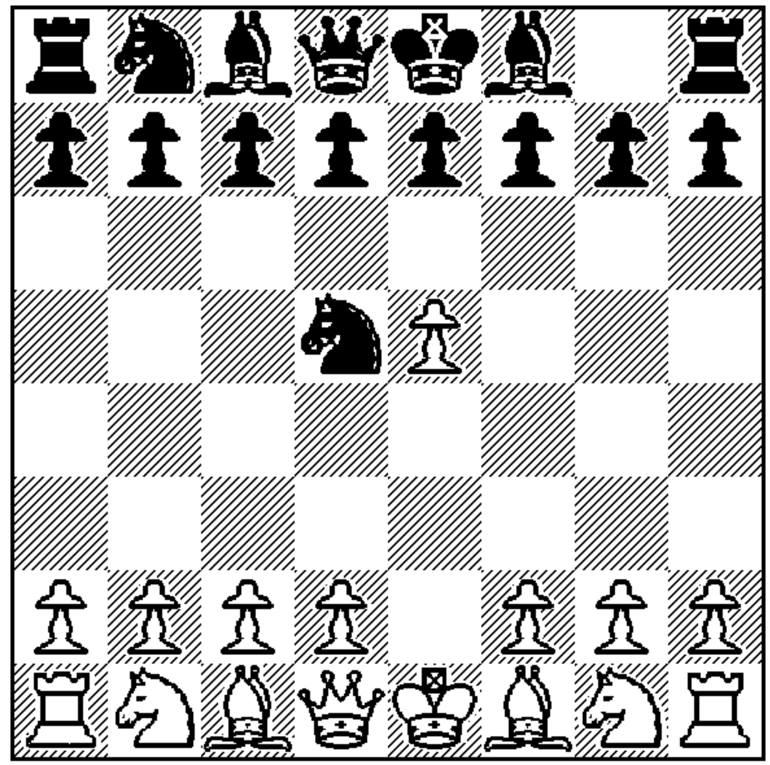

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# Motivation

- Probability theory started as a theory of games.
- Kolmogorov gave probability theory a solid axiomatic foundation.
- At the same time subjective and frequential interpretations started forming two branches of interpretations.
- The Dutch book theorem was established by Ramsay and later independently by de Finetti.
- Their interpretations were subjective. Savage and others made more subjective versions of the Dutch book theorem.
- First the payoff was money, then value, then preferences were used, and now procedures.
- Here we shall go in a less subjective direction. The payoff will be a combinatorial game.

# Combinatorial games

- Examples: Chess, nine-mens-morris, nim, go.
- The players alternate in making a move.
- Many legal moves is good, and few legal moves is bad. Convension: *losing* is the same as *no legal move*.
- We will call the players *Left* and *Right*.
- A game $G$ is specified by the options of Left $G^L$ and the options of Right $G^R$. We write

$$G = \langle G^L \mid G^R \rangle.$$

- Games are defined by *recursion* starting with the definition of the game

$$0 = \langle \emptyset \mid \emptyset \rangle.$$

# The status and group structure of a games

- If $G$ and $H$ are games then $G+H$ is the game where G and H are played in parallel.
- If $G$ is a game then $-G$ is the game where Left and Right swiches roles.
- Games have the structure of a partially ordered group.

| | |
|---|---|
| $G=0$ | if second player wins |
| $G>0$ | if Left wins whoever plays first. |
| $G<0$ | If Right wins whoever plays first. |
| $G \parallel 0$ | if first player wins. |

# All real numbers are games

- There is a way of identifying real numbers with games.

rational numbers →ordered field

→surreal numbers→games.

- There are many infinitisimal surreal numbers and many that are infinite.
- Consider social games with surreal numbers as payoffs or with games as payoffs.

# Surreal probabilities

**Two-person zero-sum games**

Let $A$ and $B$ denote finite sets and let $(a,b)\rightarrow g(a,b)$ denote a payoff function with values in an totally ordered field $F$. Then the two-person zero-sum game with payoff function g has a Nash equilibrium where the mixed strategies are probability vectors with weights in the field $F$.

**Dutch Book Theorem** Let $A$ and $B$ denote finite sets and let $(a,b)\rightarrow g(a,b)$ denote a surreal valued payoff function. Then either we have incoherence, i.e. there exist non-negative surreal numbers $q_b$ such that $\sum_{b\in B} q_b = 1$

$$\sum_{b\in B} q_b g(a,b) < 0 \text{ for all } a \in A.$$

or there exists non-negative surreal numbers $p_a$ such that $\sum_{a\in A} p_a = 1$ and

$$\sum_{a\in A} p_a g(a,b) \geq 0 \text{ for all } b \in B.$$

# Complications due to confused games

- Due to the existence of games confused with the game 0 the Dutch Book theorem becomes more complicated if we consider more general game-valued payoffs.

Theorem If a payoff function $G(a,b), a \in A, b \in B$ with short games as values, is coherent then either exists a probability vector $a \to p_a$ and a natural number n such that $np_a \in$ N and the game

$$\sum_a (np_a) \cdot G(a,b) > 0$$

for all $b \in B$, positive mean or there exist natural numbers $n_1, n_2, \cdots, n_k$, a natural number $n$ and a probability vector $a \to p_a$ such that (*) have mean value 0.

- Visit my poster if you want to understand this situation in more detail!

# Conclusion

- Frequential probabilities are *real number* but the Dutch book argument may lead to *surreal probabilities*.
- The Dutch book argument does *not* favor a subjective interpretation of probabilities.
- If the payoffs are games then *coherence* does *not imply* the *existence of a distribution* such that the mean payoff is non-negative.