Closure of independencies under graphoid properties: some experimental results

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Introducing Myself

Giuseppe Busanello

- In 2004, computer science engineering degree at University of Rome “La Sapienza”;
- In 2008, mathematics and computer science Ph. D. at University of Perugia;
- Since December 2009, Post-Doc fellowship at University of Rome “La Sapienza”;
- Main research fields
  - information measures (G. Coletti, B. Vantaggi): starting from a new characterization of a coherent conditional information measure we gave a new conditional independence definition;
  - graphical models (M. Baioletti, B. Vantaggi): a new algorithm to compute the closure with respect to graphoids and some properties for the representability by a DAG of an independence model.

Outline

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Graphoid Properties: classical context

Semi-graphoids (Dawid '79)

Let $\tilde{S}$ be a set of random variables and Y_A, Y_B, Y_C, Y_D distinct random vectors of $\tilde{S}$.
Then the independence model induced by P under classical definition is closed with respect to

G1 Symmetry
$Y_A \perp\!\!\!\perp Y_B | Y_C \Leftrightarrow Y_B \perp\!\!\!\perp Y_A | Y_C$

G2 Decomposition
$Y_A \perp\!\!\!\perp [Y_B, Y_C] | Y_D \Rightarrow Y_A \perp\!\!\!\perp Y_B | Y_D$ and $Y_A \perp\!\!\!\perp Y_C | Y_D$

G3 Weak Union
$Y_A \perp\!\!\!\perp [Y_B, Y_C] | Y_D \Rightarrow Y_A \perp\!\!\!\perp Y_B | [Y_C, Y_D]$

G4 Contraction
$Y_A \perp\!\!\!\perp Y_B | Y_D$ and $Y_A \perp\!\!\!\perp Y_C | [Y_B, Y_D] \Rightarrow Y_A \perp\!\!\!\perp [Y_B, Y_C] | Y_D$

G5 Intersection if P strictly positive
$Y_A \perp\!\!\!\perp Y_B | [Y_C, Y_D]$ and $Y_A \perp\!\!\!\perp Y_C | [Y_B, Y_D] \Rightarrow Y_A \perp\!\!\!\perp [Y_B, Y_C] | [Y_D]$

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Graphoid Properties: cs-independence

Graphoids (Vantaggi 2001)

- The independence model induced by cs-independence (introduced in Coletti-Scozzafava 2000) is not necessarily closed with respect to symmetry.
- By means of a reinforcement that essentially requires the symmetry, it is closed with respect to graphoids G1–G5.

Notation

- $\tilde{S} = \{Y_1, ..., Y_n\}$ is a finite set of random variables, with $S = \{1, ..., n\}$;
- (A, B, C) corresponds to $Y_A \perp\!\!\!\perp Y_B | Y_C$ with $X = (A \cup B \cup C)$ and J is a nonempty subset of independence relations compatible with a given (coherent conditional) probability.
- $\overline{J}$ is the closure of J with respect to graphoids.

Outline

Definition

Definition (g-inclusion)

Given a pair of triples $\theta = (A, B, C)$, $\theta' = (A', B', C')$ then θ' is generalized-included to θ ($\theta' \sqsubseteq \theta$) if and only if

- $C \subseteq C' \subseteq A \cup B \cup C$ *and*
- $[A' \subseteq A$ *and* $B' \subseteq B]$ *or* $[B' \subseteq A$ *and* $A' \subseteq B]$.

Any θ' g-included to θ can be obtained from θ with a finite number of applications of symmetry, decomposition and weak-union properties.

The g–inclusion is the symmetrize version of *dominance* relation (Studený '97).

Main Goal

$$J_* = \{\theta \in \overline{J} : \text{ for each } \tau \in \overline{J}, \tau \neq \theta, \theta \not\sqsubseteq \tau\}$$

J_* has the same information of $\overline{J}$ but in general J_* is smaller than $\overline{J}$.
J_* is called *fast closure*.
Any triple θ of J_* is called *maximal triple*.

Outline

Basis Idea

Maximal triples of a pair of triples

Given a pair of triples θ_1, θ_2, that is

$\theta_1 = (A_A \cup A_B \cup A_C \cup A_N, B_A \cup B_B \cup B_C \cup B_N, C_A \cup C_B \cup C_C \cup C_N)$;

$\theta_2 = (A_A \cup B_A \cup C_A \cup A'_N, A_B \cup B_B \cup C_B \cup B'_N, A_C \cup B_C \cup C_C \cup C'_N)$;

then $J_* = \{\theta_1, \theta_2\}_*$ is a subset of the set $K(\theta_1, \theta_2)$ composed by θ_1, θ_2 and

1. $(A_A, A_B \cup B_A \cup B_B \cup B_C \cup C_B \cup B_N, A_C \cup C_A \cup C_C)$;
2. $(A_A, A_B \cup B_A \cup B_B \cup B_C \cup C_B \cup B'_N, A_C \cup C_A \cup C_C)$;
3. $(A_B, A_A \cup B_A \cup B_B \cup B_C \cup C_A \cup B_N, A_C \cup C_B \cup C_C)$;
4. $(A_B, A_A \cup B_A \cup B_B \cup B_C \cup C_A \cup A'_N, A_C \cup C_B \cup C_C)$;
5. $(B_A, A_A \cup A_B \cup A_C \cup B_B \cup C_B \cup A_N, B_C \cup C_A \cup C_C)$;
6. $(B_A, A_A \cup A_B \cup A_C \cup B_B \cup C_B \cup B'_N, B_C \cup C_A \cup C_C)$;
7. $(B_B, A_A \cup A_B \cup A_C \cup B_A \cup C_A \cup A_N, B_C \cup C_B \cup C_C)$;
8. $(B_B, A_A \cup A_B \cup A_C \cup B_A \cup C_A \cup A'_N, B_C \cup C_B \cup C_C)$;
9. $(A_B \cup B_A, A_A \cup B_B, A_C \cup B_C \cup C_A \cup C_B \cup C_C)$.

From the previous procedure one inferential rule arises for computing J_*.

function FC1(J)
begin
$J_0 := N_0 := J$
$k := 0$
repeat
$k := k + 1$
$N_k := \{\tau : \tau \in \{\theta_1, \theta_2\}_* \text{ with } \theta_1 \in J_{k-1}, \theta_2 \in N_{k-1}\}$
$J_k := \text{FINDMAXIMAL}(J_{k-1} \cup N_k)$
until $J_k = J_{k-1}$
return J_k
end

For each J subset of $S^{(3)}$ then FC1$(J) = J_*$.

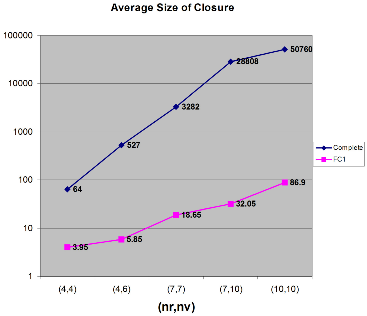
Experiments

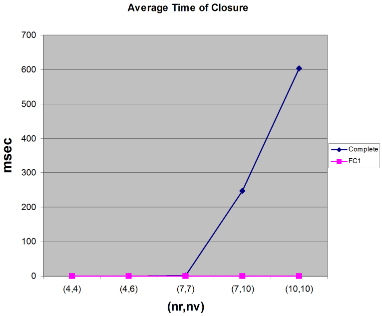
Technical data

- **CPU:** AMD Dual Core Opteron at 1.8 GHz
- **RAM:** 2 GByte
- **O.S.:** Linux
- **Cut-Off:** 5,000,000 of triples (only for FC1)
- **Time-out:** 3600 seconds
- **(nr, nv):** nr = number of relations, nv = number of random variables

Performance Experiment

Classical closure Vs. FC1: size and time closures





Open Problems

- Find a suitable data structure for representing the triples.
- Find if there are relationships among triples to reduce the number of useless generated triples.
- Test if the implication problem can be solved in a faster way.
- Find an optimal graph representing the triples of J_*.

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