Affinity and Continuity of Credal Set Operator

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Motivation

The study of **geometrical relations** between two classes of imprecise probability models of Walley:

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Application:

- any co-co set of CLPs is described by K-M theorem
- characterize the image of this set via credal set operator

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$\mathcal{P}$	weak*-compact convex set in $\mathscr{L}^*$ of all linear previsions on $\mathscr{L}$
$\mathcal{M}(\underline{P})$	credal set of $\underline{P}$ $= \{P \in \mathcal{P} \mid P(f) \geq \underline{P}(f), f \in \mathscr{K}\}$
$E_{\underline{P}}$	natural extension of $\underline{P}$ to $\mathscr{L}$ $= \inf_{P \in \mathcal{M}(\underline{P})} P$

Credal Set Operator

$\mathscr{K}$	a set of gambles
$\underline{\mathcal{C}}_{\mathscr{K}}$	the convex set of all CLPs on $\mathscr{K}$
S	the set of all nonempty weak*-compact convex subsets of $\mathcal{P}$

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Does $\mathcal{M}$ preserve the operations and the structure from $\underline{\mathcal{C}}_{\mathscr{K}}$ to S?

- **algebraic** structure
- **topological** structure

Algebraic Operations with Credal Sets

The set $2^{\mathscr{L}^*}$ endowed with the Minkowski sum $\oplus$ and the scalar multiplication is a **real semilinear space**:

- $\left(2^{\mathscr{L}^*}, \oplus, \{0\}\right)$ is a commutative monoid
- $\alpha(\beta\mathcal{A}) = (\alpha\beta)\mathcal{A} \quad \forall\alpha, \beta \in \mathbb{R}, \forall\mathcal{A} \in 2^{\mathscr{L}^*}$
- $1\mathcal{A} = \mathcal{A},\ 0\mathcal{A} = \{0\}$
- $\alpha(\mathcal{A}_1 \oplus \mathcal{A}_2) = (\alpha\mathcal{A}_1) \oplus (\alpha\mathcal{A}_2), \quad \forall\mathcal{A}_1, \mathcal{A}_2 \in 2^{\mathscr{L}^*}, \alpha \in \mathbb{R}$

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The set S is **convex** in $2^{\mathscr{L}^*}$:

$$\alpha\mathcal{A}_1 \oplus (1-\alpha)\mathcal{A}_2 \in \mathsf{S}, \quad \forall \mathcal{A}_1, \mathcal{A}_2 \in \mathsf{S}, \alpha \in [0,1]$$

Representation of Credal Sets by Superdifferentials

The **superdifferential** of a concave function $E_{\underline{P}}$ at $f \in \mathscr{L}$ is the set

$$\partial E_{\underline{P}}(f) = \{P^* \in \mathscr{L}^* \mid P^* \geq \mathrm{d}^+ E_{\underline{P}}(f)(.)\},$$

where for $g \in \mathscr{L}$,

$$\mathrm{d}^+ E_{\underline{P}}(f)(g) = \lim_{t \to 0_+} \frac{E_{\underline{P}}(f + tg) - E_{\underline{P}}(f)}{t}.$$

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Theorem

If $\underline{P}$ is a coherent lower prevision on a set of gambles $\mathscr{K}$ and $E_{\underline{P}}$ is the natural extension of $\underline{P}$, then

$$\mathcal{M}(\underline{P}) = \partial E_{\underline{P}}(1).$$

Affinity

A mapping $\Gamma : \underline{\mathcal{C}}_{\mathscr{K}} \to \mathsf{S}$ is **affine** if for every $\underline{P}^1, \ldots, \underline{P}^n \in \underline{\mathcal{C}}_{\mathscr{K}}$ and $\alpha_i \in [0, 1], i = 1, \ldots, n$, with $\sum_{i=1}^n \alpha_i = 1$,

$$\Gamma\left(\sum_{i=1}^n \alpha_i \underline{P}^i\right) = \bigoplus_{i=1}^n \alpha_i \Gamma(\underline{P}^i).$$

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If the mapping

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Theorem

The credal set operator $\mathcal{M}$ *is affine on the convex set of all supermodular coherent lower probabilities on* 2^Ω*.*

Belief Functions

Theorem

Let Ω *be finite,* $\underline{P}$ *be a* **belief measure** *on* 2^Ω *and* $\mu^{\underline{P}}$ *its* **Möbius transform**. *Then*

$$\mathcal{M}(\underline{P}) = \bigoplus_{A \subseteq \Omega} \mu^{\underline{P}}(A) \mathcal{S}_A,$$

where $\mathcal{S}_A = \{P \in \mathcal{P} \mid P(A) = 1\}$.

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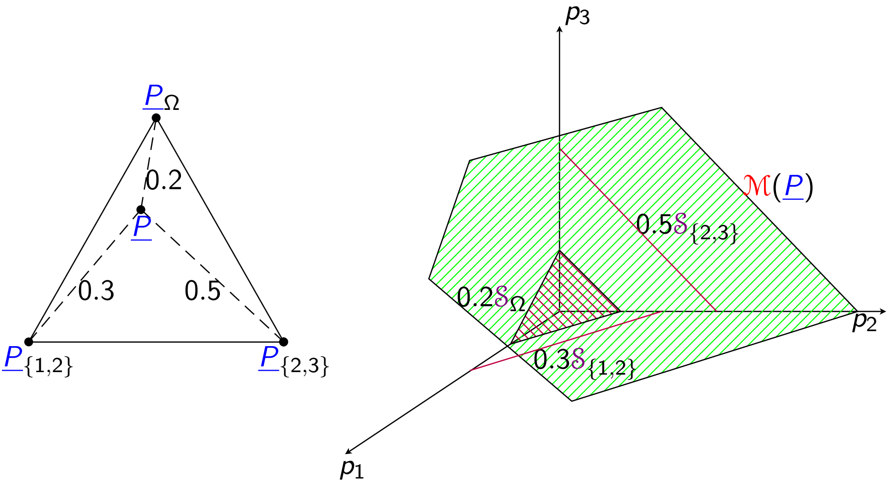
Example

Let $\Omega = \{1, 2, 3\}$ and $\underline{P}$ be a belief measure whose Möbius transform $\mu^{\underline{P}}$ is

$$\mu^{\underline{P}}(A) = \begin{cases} 0.2, & A = \Omega, \\ 0.3, & A = \{1, 2\}, \\ 0.5, & A = \{2, 3\}, \\ 0, & \textit{otherwise.} \end{cases}$$

Belief Functions (ctnd.)



Continuity

Assumption $|\Omega| = n$

Theorem

If S *is endowed with the topology of* **Hausdorff metric**, *then* $\mathcal{M} : \underline{\mathcal{C}}_{\mathscr{L}} \to \mathsf{S}$ *is an affine homeomorphism.*

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Theorem

If $\underline{P} \in \underline{\mathcal{C}}_{\mathscr{K}}$, *then there exists a sequence* $(\mathcal{S}_n)$ *of* **simple polytopes** *in* S *such that*

- $\mathcal{S}_n \to \mathcal{M}(\underline{P})$ *in the Hausdorff metric*
- $\mathcal{M}^{-1}(\mathcal{S}_n) \to \underline{P}$ *pointwise on* $\mathscr{K}$
- $\mathcal{M}^{-1}(\mathcal{S}_n) \to \underline{P}$ *uniformly on each compact subset of* $\mathscr{K}$