Multivariate Models and Confidence Intervals: A Local Random Set Approach

Thomas Fetz

Unit for Engineering Mathematics
University of Innsbruck, Austria
Thomas.Fetz@uibk.ac.at

6th International Symposium on Imprecise Probability: Theories and Applications, Durham, United Kingdom, 2009

Research Team (imprecise probabilities):

- Michael Oberguggenberger (head of unit)
- Bernhard Schmelzer (next presentation)
- Thomas Fetz

My research area:

- Propagating uncertainty through a mapping.
- Imprecise probabilities and independence.
- Starting: Bayesian Networks linked with Geographic Information Systems (GIS) in collaboration with civil engineers of our faculty.

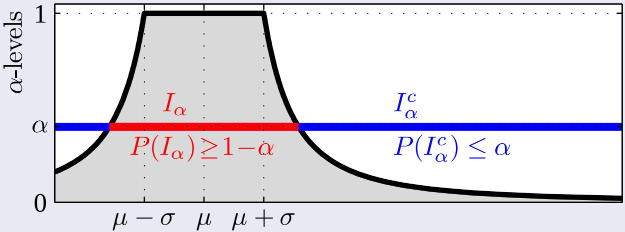
Starting point: Non-parametric models, Tchebycheff

Given: Variable X with $\mu = \mathrm{E}(X)$ and $\sigma^2 = \mathrm{V}(X)$ as sole information.

Generating a nested family $\mathbf{I} = \{I_\alpha\}_{\alpha \in (0,1]}$ of confidence intervals

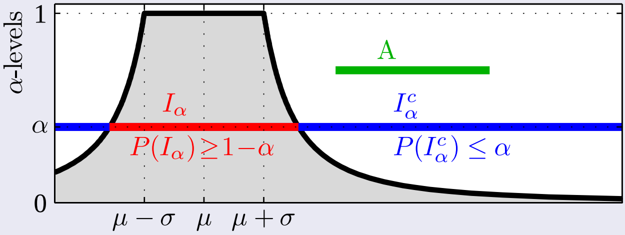
$$I_\alpha = \left[\mu - \frac{\sigma}{\sqrt{\alpha}}, \mu + \frac{\sigma}{\sqrt{\alpha}}\right], \quad \alpha \in (0,1]$$

using Tchebycheff's inequality $P(|X-\mu| > \frac{\sigma}{\sqrt{\alpha}}) \leq \alpha$.



Questions

- What is it? Is it a random set or a fuzzy set?
- What is the upper probability $\overline{P}(A)$ of an event A?
  What is its interpretation with respect to confidence intervals?
- What happens in the multivariate case?

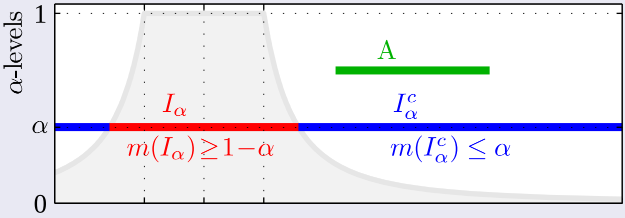


Looking at a single confidence interval

Equipping the two intervals I_α and I_α^c with weights

$$m(I_\alpha) = P(I_\alpha) \text{ and } m(I_\alpha^c) = P(I_\alpha^c)$$

we get a local random set at level α.



Looking at a single confidence interval

Equipping the two sets I_α and I_α^c with weights

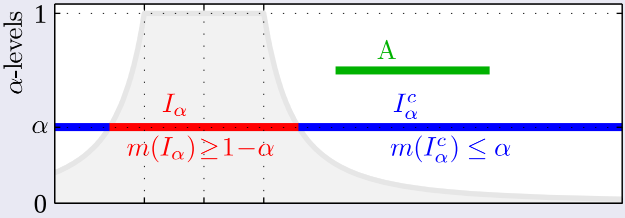
$$m(I_\alpha) = P(I_\alpha) \quad \text{and} \quad m(I_\alpha^c) = P(I_\alpha^c)$$

we get a local random set at level α.

The local upper probability $\overline{P}_\alpha(A)$ at level α for an event A is

$$\overline{P}_\alpha(A) = m(I_\alpha)\,\chi(A \cap I_\alpha \neq \varnothing) + m(I_\alpha^c)\,\chi(A \cap I_\alpha^c \neq \varnothing).$$

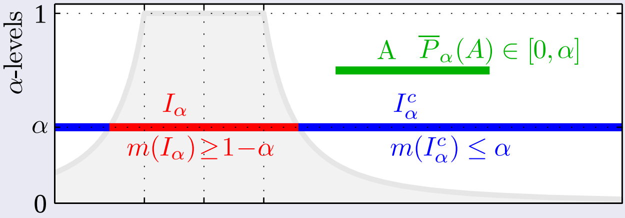
(χ indicator function)



Three different cases for an event A

(i)	$A \cap I_\alpha = \varnothing$	$\overline{P}_\alpha(A) \in [0, \alpha]$
(ii)	$A \cap I_\alpha^c = \varnothing$	$\overline{P}_\alpha(A) \in [1-\alpha, 1]$
(iii)	$A \cap I_\alpha \neq \varnothing$ and $A \cap I_\alpha^c \neq \varnothing$	$\overline{P}_\alpha(A) = 1$

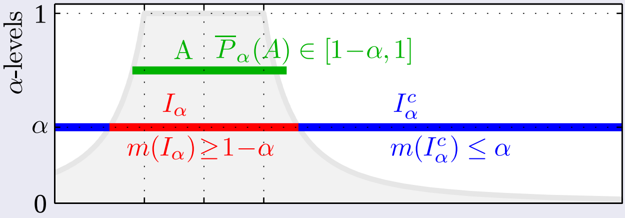
$$\overline{P}_\alpha(A) = m(I_\alpha)\,\chi(A \cap I_\alpha \neq \varnothing) + m(I_\alpha^c)\,\chi(A \cap I_\alpha^c \neq \varnothing).$$



Three different cases for an event A

(i)	$A \cap I_\alpha = \varnothing$	$\overline{P}_\alpha(A) \in [0, \alpha]$
(ii)	$A \cap I_\alpha^c = \varnothing$	$\overline{P}_\alpha(A) \in [1-\alpha, 1]$
(iii)	$A \cap I_\alpha \neq \varnothing$ and $A \cap I_\alpha^c \neq \varnothing$	$\overline{P}_\alpha(A) = 1$

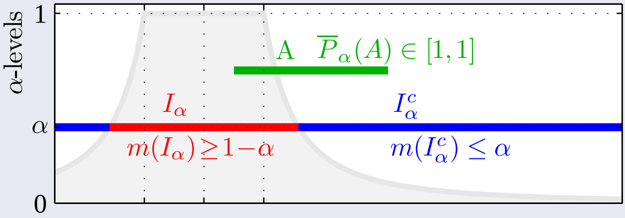
$$\overline{P}_\alpha(A) = m(I_\alpha)\,\chi(A \cap I_\alpha \neq \varnothing) + m(I_\alpha^c)\,\chi(A \cap I_\alpha^c \neq \varnothing).$$



Three different cases for an event A

(i)	$A \cap I_\alpha = \varnothing$	$\overline{P}_\alpha(A) \in [0, \alpha]$
(ii)	$A \cap I_\alpha^c = \varnothing$	$\overline{P}_\alpha(A) \in [1-\alpha, 1]$
(iii)	$A \cap I_\alpha \neq \varnothing$ and $A \cap I_\alpha^c \neq \varnothing$	$\overline{P}_\alpha(A) = 1$

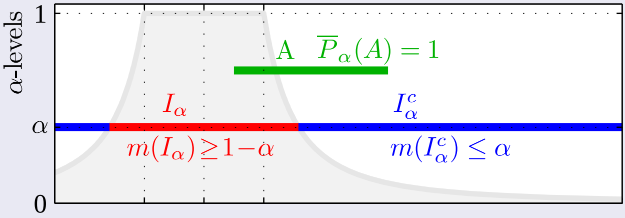
$$\overline{P}_\alpha(A) = m(I_\alpha)\,\chi(A \cap I_\alpha \neq \varnothing) + m(I_\alpha^c)\,\chi(A \cap I_\alpha^c \neq \varnothing).$$



Three different cases for an event A

(i)	$A \cap I_\alpha = \varnothing$	$\overline{P}_\alpha(A) = \alpha$
(ii)	$A \cap I_\alpha^c = \varnothing$	$\overline{P}_\alpha(A) = 1$
(iii)	$A \cap I_\alpha \neq \varnothing$ and $A \cap I_\alpha^c \neq \varnothing$	$\overline{P}_\alpha(A) = 1$

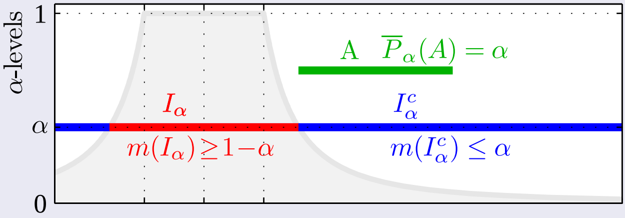
To avoid interval-valued $\overline{P}_\alpha(A)$ we take always the upper bounds.



Most interesting case (i)

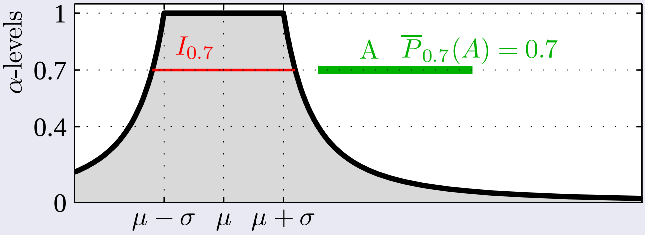
- A has the role of the “bad and undesired” event.
- Meaning:

If A is outside the confidence interval I_α at confidence level $1-\alpha$, then we can say for sure that A occurs only with probability α, at most.



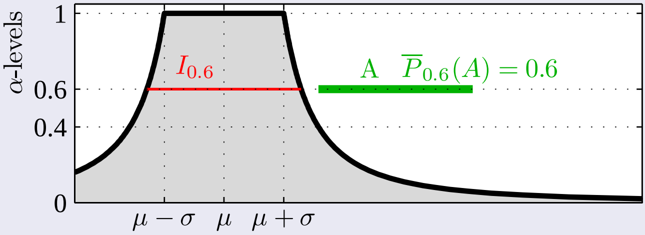
Formula for the upper probability $\overline{P}(A)$

$$\overline{P}(A) = \inf_{\alpha\in(0,1]} \overline{P}_\alpha(A) = \inf_{\alpha\in(0,1]} \{\alpha : I_\alpha \cap A = \varnothing\}$$



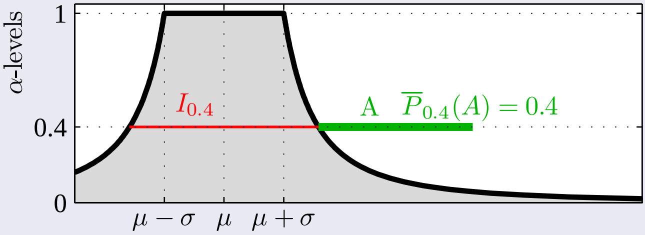
Formula for the upper probability $\overline{P}(A)$

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \overline{P}_\alpha(A) = \inf_{\alpha \in (0,1]} \{\alpha : I_\alpha \cap A = \varnothing\}$$



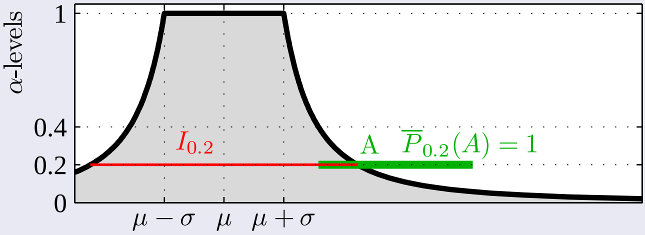
Formula for the upper probability $\overline{P}(A)$

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \overline{P}_\alpha(A) = \inf_{\alpha \in (0,1]} \{\alpha : I_\alpha \cap A = \varnothing\}$$



Formula for the upper probability $\overline{P}(A)$

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \overline{P}_\alpha(A) = \inf_{\alpha \in (0,1]} \{\alpha : I_\alpha \cap A = \emptyset\}$$

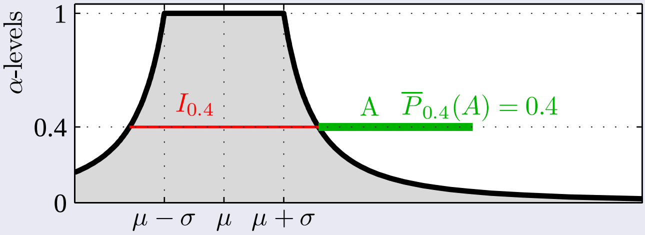


Formula for the upper probability $\overline{P}(A)$

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \overline{P}_\alpha(A) = \inf_{\alpha \in (0,1]} \{\alpha : I_\alpha \cap A = \emptyset\}$$

Interpretation of $\mathbf{I}$ as random set or fuzzy set

$$\overline{P}(A) = \text{Plausibility}(A) = \text{Possibility}(A)$$



Goal

A formula for the upper probability for given families $\mathbf{I}_1, \ldots, \mathbf{I}_n$ of confidence intervals similar to the univariate version.

Goal

A formula for the upper probability for given families $\mathbf{I}_1, \ldots, \mathbf{I}_n$ of confidence intervals similar to the univariate version.

Short preview

univariate

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \{\alpha : I_\alpha \cap A = \varnothing\}$$

multivariate

$$\overline{P}_\ell^S(A) = \inf_{\boldsymbol{\alpha} \in S} \{\ell(\boldsymbol{\alpha}) : J_{\boldsymbol{\alpha}} \cap A = \varnothing\}$$

Goal

A formula for the upper probability for given families $\mathbf{I}_1, \ldots, \mathbf{I}_n$ of confidence intervals similar to the univariate version.

Short preview

univariate

$$\overline{P}(A) = \inf_{\alpha \in (0,1]} \{\alpha : I_\alpha \cap A = \emptyset\}$$

multivariate

$$\overline{P}_\ell^S(A) = \inf_{\boldsymbol{\alpha} \in S} \{\ell(\boldsymbol{\alpha}) : J_{\boldsymbol{\alpha}} \cap A = \emptyset\}$$

Possibilities of choice

1. For the set of confidence intervals considered to be combined.
2. For the weights used for the local joint random set.

Combination of marginal confidence intervals

$\mathbf{J} = \{J_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in S}$ is the family of all joint confidence sets

$$J_{\boldsymbol{\alpha}} = I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}, \quad I_{k,\alpha_k} \in \mathbf{I}_k$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ depending on the set S of indices $\boldsymbol{\alpha}$:

1. Random set independence like: $S = S_{\mathrm{R}} = (0,1]^n$.
2. Fuzzy set independence like: $S = S_{\mathrm{F}} = \{\boldsymbol{\alpha} \in (0,1]^n : \alpha_1 = \cdots = \alpha_n\}$

Combination of marginal confidence intervals

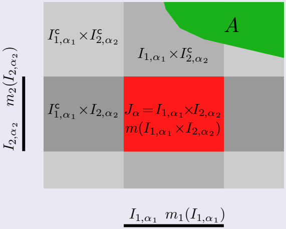
$\mathbf{J} = \{J_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in S}$ is the family of all joint confidence sets

$$J_{\boldsymbol{\alpha}} = I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}, \quad I_{k,\alpha_k} \in \mathbf{I}_k$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ depending on the set S of indices $\boldsymbol{\alpha}$:

1. Random set independence like: $S = S_{\mathrm{R}} = (0,1]^n$.
2. Fuzzy set independence like: $S = S_{\mathrm{F}} = \{\boldsymbol{\alpha} \in (0,1]^n : \alpha_1 = \cdots = \alpha_n\}$

Local upper probability



Event A with $J_{\boldsymbol{\alpha}} \cap A = \varnothing$.

Combination of marginal confidence intervals

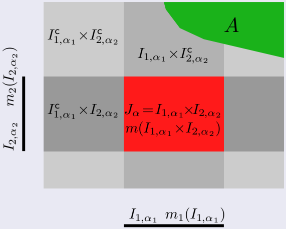
$\mathbf{J} = \{J_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in S}$ is the family of all joint confidence sets

$$J_{\boldsymbol{\alpha}} = I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}, \quad I_{k,\alpha_k} \in \mathbf{I}_k$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ depending on the set S of indices $\boldsymbol{\alpha}$:

1. Random set independence like: $S = S_{\mathrm{R}} = (0,1]^n$.
2. Fuzzy set independence like: $S = S_{\mathrm{F}} = \{\boldsymbol{\alpha} \in (0,1]^n : \alpha_1 = \cdots = \alpha_n\}$

Local upper probability



Event A with $J_{\boldsymbol{\alpha}} \cap A = \varnothing$.

$$\overline{P}_{\boldsymbol{\alpha}}(A) \leq \overline{P}((I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})^c) = \\ = 1 - m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}).$$

Combination of marginal confidence intervals

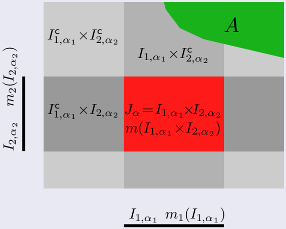
$\mathbf{J} = \{J_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in S}$ is the family of all joint confidence sets

$$J_{\boldsymbol{\alpha}} = I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}, \quad I_{k,\alpha_k} \in \mathbf{I}_k$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ depending on the set S of indices $\boldsymbol{\alpha}$:

1. Random set independence like: $S = S_{\mathrm{R}} = (0,1]^n$.
2. Fuzzy set independence like: $S = S_{\mathrm{F}} = \{\boldsymbol{\alpha} \in (0,1]^n : \alpha_1 = \cdots = \alpha_n\}$

Local upper probability



Event A with $J_{\boldsymbol{\alpha}} \cap A = \varnothing$.

$$\overline{P}_{\boldsymbol{\alpha}}(A) = \overline{P}((I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})^c) = 1 - m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}).$$

Worst case.

Random set independence

Product of the marginal weights: $m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) = \prod_{i=1}^{n} m_i(I_{i,\alpha_i})$.

Leads to $$\boxed{\overline{P}_{\boldsymbol{\alpha}}(A) = 1 - \prod_{i=1}^{n}(1-\alpha_i).}$$

- Used if the uncertain variables are independent.

Random set independence

Product of the marginal weights: $m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) = \prod_{i=1}^{n} m_i(I_{i,\alpha_i})$.

Leads to $$\overline{P}_{\boldsymbol{\alpha}}(A) = 1 - \prod_{i=1}^{n}(1-\alpha_i).$$

- Used if the uncertain variables are independent.

Lower / upper bounds

Lower and upper bounds of Fréchet for the joint weights:

$$\max\Big(\sum_{i=1}^{n} m(I_{i,\alpha_i}) - n + 1, 0\Big) \le m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) \le \min_{i=1,\ldots,n} m(I_{i,\alpha_i}).$$

Leads to $$\max_{i=1,\ldots,n}(\alpha_i) \le \overline{P}_{\boldsymbol{\alpha}}(A) \le \min(\alpha_1 + \cdots + \alpha_n, 1).$$

- Used if nothing is known about interactions between the variables.

Level function $\ell(\alpha)$

The different approaches have only an influence on the level

$$\ell(\boldsymbol{\alpha}) = \begin{cases} \max\limits_{i=1,\dots,n} (\alpha_i) & \text{lower bound,} \\ 1 - \prod_{i=1}^{n}(1-\alpha_i) & \text{random set independence,} \\ \min(\alpha_1 + \cdots + \alpha_n, 1) & \text{upper bound} \end{cases}$$

of the joint confidence set $J_{\boldsymbol{\alpha}}$, but not on the confidence set itself.

Level function $\ell(\alpha)$

The different approaches have only an influence on the level

$$\ell(\boldsymbol{\alpha}) = \begin{cases} \max\limits_{i=1,\dots,n} (\alpha_i) & \text{lower bound,} \\ 1 - \prod_{i=1}^{n}(1-\alpha_i) & \text{random set independence,} \\ \min(\alpha_1 + \dots + \alpha_n, 1) & \text{upper bound} \end{cases}$$

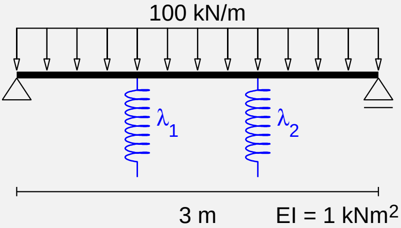
of the joint confidence set $J_{\boldsymbol{\alpha}}$, but not on the confidence set itself.

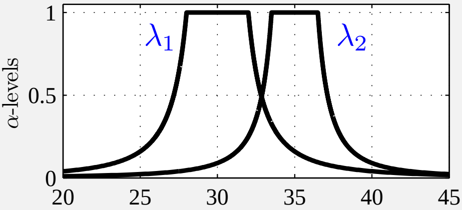
Formula for the upper probability similar to the univariate case

$$\boxed{\overline{P}_\ell^S(A) = \inf_{\boldsymbol{\alpha}\in S}\{\ell(\boldsymbol{\alpha}) : J_{\boldsymbol{\alpha}} \cap A = \varnothing\}.}$$

Numerical Example

Beam bedded on two springs with uncertain spring constants





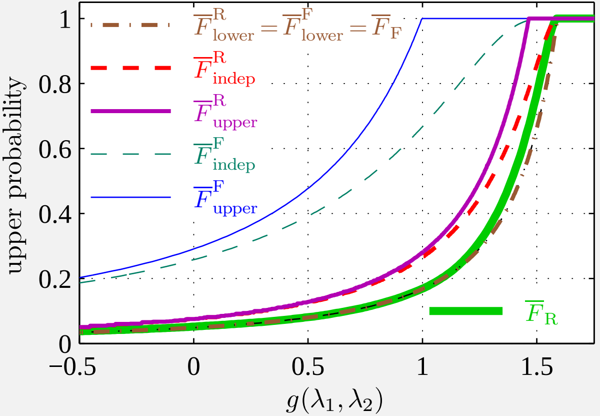
Criterion of failure of beam: $g(\lambda_1, \lambda_2) \leq 0$

Failure function

$$g(\lambda_1, \lambda_2) = M_{\text{yield}} - \max_{x \in [0,3]} |M(x, \lambda_1, \lambda_2)|$$

- $M(x, \lambda_1, \lambda_2)$ is the bending moment at x depending on λ_1, λ_2.
- $M_{\text{yield}} = 12$ kNm is the elastic limit moment.

Upper probability distributions $\overline{F}^S_\ell(g(\lambda_1, \lambda_2))$



The upper probabilities of failure are the results at $g(\lambda_1, \lambda_2) = 0$.

Ordering of the upper probabilities

$$\overline{P}_{\mathrm{F}}(A) = \overline{P}^{\mathrm{R}}_{\mathrm{lower}}(A) = \overline{P}^{\mathrm{F}}_{\mathrm{lower}}(A)$$

$$\overline{P}_{\mathrm{R}}(A) \leq \overline{P}^{\mathrm{R}}_{\mathrm{indep}}(A) \leq \overline{P}^{\mathrm{F}}_{\mathrm{indep}}(A)$$

$$\overline{P}_{\mathrm{U}}(A) \leq \overline{P}^{\mathrm{R}}_{\mathrm{upper}}(A) \leq \overline{P}^{\mathrm{F}}_{\mathrm{upper}}(A)$$

$\downarrow$

Classical approaches

Notation	
$\overline{P}_{\mathrm{F}}$	fuzzy set independence
$\overline{P}_{\mathrm{R}}$	random set independence
$\overline{P}_{\mathrm{U}}$	unknown interaction / Fréchet

Ordering of the upper probabilities

$$\overline{P}_{\mathrm{F}}(A) = \overline{P}^{\mathrm{R}}_{\mathrm{lower}}(A) = \overline{P}^{\mathrm{F}}_{\mathrm{lower}}(A)$$

$$\overline{P}_{\mathrm{R}}(A) \leq \overline{P}^{\mathrm{R}}_{\mathrm{indep}}(A) \leq \overline{P}^{\mathrm{F}}_{\mathrm{indep}}(A)$$

$$\overline{P}_{\mathrm{U}}(A) \leq \overline{P}^{\mathrm{R}}_{\mathrm{upper}}(A) \leq \overline{P}^{\mathrm{F}}_{\mathrm{upper}}(A)$$

$\downarrow$

All possible combinations of confidence intervals, $S = S_{\mathrm{R}}$

Notation	level $\ell(\boldsymbol{\alpha})$	
$\overline{P}^{\mathrm{R}}_{\mathrm{lower}}$	$\max_{i=1,\ldots,n}(\alpha_i)$	lower Fréchet bound
$\overline{P}^{\mathrm{R}}_{\mathrm{indep}}$	$1 - \prod_{i=1}^{n}(1-\alpha_i)$	random set independence
$\overline{P}^{\mathrm{R}}_{\mathrm{upper}}$	$\min\left(\sum_{i=1}^{n}\alpha_i, 1\right)$	upper Fréchet bound

Ordering of the upper probabilities

$$\overline{P}_{\mathrm{F}}(A) = \overline{P}^{\mathrm{R}}_{\mathrm{lower}}(A) = \overline{P}^{\mathrm{F}}_{\mathrm{lower}}(A)$$

$$\overline{P}_{\mathrm{R}}(A) \leq \overline{P}^{\mathrm{R}}_{\mathrm{indep}}(A) \leq \overline{P}^{\mathrm{F}}_{\mathrm{indep}}(A)$$

$$\overline{P}_{\mathrm{U}}(A) \leq \overline{P}^{\mathrm{R}}_{\mathrm{upper}}(A) \leq \overline{P}^{\mathrm{F}}_{\mathrm{upper}}(A)$$

$\downarrow$

Combinations of intervals of the same level α only, $S = S_{\mathrm{F}}$

Notation	level $\ell(\alpha)$	
$\overline{P}^{\mathrm{F}}_{\mathrm{lower}}$	α	lower Fréchet bound
$\overline{P}^{\mathrm{F}}_{\mathrm{indep}}$	$1-(1-\alpha)^n$	random set independence
$\overline{P}^{\mathrm{F}}_{\mathrm{upper}}$	$\min(n\alpha, 1)$	upper Fréchet bound