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On general conditional random quantities

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Fitting of our contribution in the context of ISIPTA:

We study the coherence of prevision assessments for general conditional random quantities, like $X|Y$, where X and Y are finite discrete random quantities.

We generalize the study concerning:

- conditional events, like $E|H$, where E and H are events;

- conditional random quantities, like $X|H$, where X is a random quantity and H is an event.

Outline

• *We consider the notion of* general conditional prevision *of the form* $\mathbb{P}(X|Y)$, *where both* X *and* Y *are random quantities, introduced in (Lad and Dickey, 1990).*

• *We integrate the analysis of Lad and Dickey by properly managing the case* $\mathbb{P}(Y) = 0$

• *We propose a definition of coherence for the conditional prevision of '*X *given* Y*'*

• *We obtain some results on coherence of a conditional prevision assessment* $\mathbb{P}(X|Y) = \mu$ *in the finite case*

Basic notions

In the setting of coherence, given any r. q. X and any events E, H, with $P(E|H) = p$ and $\mathbb{P}(X|H) = \mu$, if you pay p (resp., μ) you receive $E|H$ (resp., $X|H$); then, *operatively*, it is

$$E|H = EH + pH^c = EH + p(1 - H)\,,$$

$$X|H = XH + \mu H^c = XH + \mu(1 - H)\,.$$

A general conditional r. q. $X|Y$ is obtained by replacing in the last formula the event H (and its indicator) by a r. q. Y.

Definition 1. (Lad & Dickey)
Given two r. q. X and Y, the conditional prevision for 'X given Y', denoted $\mathbb{P}(X|Y)$, is a number you specify with the understanding that you accept to engage any transaction yielding a random net gain $G = sY[X - \mathbb{P}(X|Y)]$.

Definition 2. (Lad & Dickey)
Having asserted your conditional prevision $\mathbb{P}(X|Y) = \mu$, the c. r. q. $X|Y$ is defined as

$$X|Y = XY + (1 - Y)\mu = \mu + Y(X - \mu)\,.$$

Then $G = sY(X - \mu) = s(X|Y - \mu)$ and, as $\mathbb{P}(G) = 0$, it follows (*generalized compound prevision theorem*)

$$\mathbb{P}(XY) = \mathbb{P}(X|Y)\mathbb{P}(Y)\,.$$

Some remarks.

1) if $Y \equiv 0$, you always receive the same amount $\mu = \mathbb{P}(X|Y)$ that you have payed (the net gain is always 0). To avoid this trivial case we will assume that $(Y = 0) \neq \Omega$.

2) if X and Y are uncorrelated, it is $\mathbb{P}(XY) = \mathbb{P}(X)\mathbb{P}(Y)$; then, assuming $\mathbb{P}(Y) \neq 0$, it follows $\mathbb{P}(X|Y) = \mathbb{P}(X)$.

In other words, *under the hypothesis* $\mathbb{P}(Y) \neq 0$, *X and Y are uncorrelated if and only if the prevision of 'X given Y' coincides with the prevision of X.*

3) $\mathbb{P}(Y) = 0 \;\not\Rightarrow\; \mathbb{P}(XY) = 0$; **then, it may happen that doesn't exist any finite value of $\mathbb{P}(X|Y)$ which satisfies the equality**

$$\mathbb{P}(XY) = \mathbb{P}(X|Y)\mathbb{P}(Y)\,.$$

A critical example

(where $\mathbb{P}(Y) = 0\,, \quad \mathbb{P}(XY) \neq 0$)

$(X, Y) \in \{(0, -1), (0, 1), (1, -1), (1, 1)\}$;
we set $p(x, y) = P(X = x, Y = y)$, with

$$p(0, -1) = \frac{1}{3}, \;\; p(0, 1) = \frac{1}{6}\,,$$
$$p(1, -1) = \frac{1}{6}, \;\; p(1, 1) = \frac{1}{3}\,.$$

We have

$$Y \in \{-1, 1\}\,, \; XY \in \{-1, 0, 1\}\,,$$

with

$$P(Y=-1)=P(Y=1)=\frac{1}{2},$$

$$P(XY=0)=\frac{1}{2},$$

$$P(XY=-1)=\frac{1}{6},\ P(XY=1)=\frac{1}{3};$$

so that $\mathbb{P}(Y)=0$ and $\mathbb{P}(XY)=\frac{1}{6}$;

hence, *the equation* $\frac{1}{6} = \mathbb{P}(X|Y)\cdot 0$ *has no solutions.*

What can be said about coherence of the assessment $\mathbb{P}(X|Y) = \mu$ when $\mathbb{P}(Y) = 0$?

To properly manage the case $\mathbb{P}(Y) = 0$, we integrate the work of Lad and Dickey
(i) by using an explicit definition of coherence for any given assessment $\mathbb{P}(X|Y) = \mu$;
(ii) by discarding, in the definition of coherence, the value 0 of the net gain associated with the case $Y = 0$.

Definition of coherence. Given two r. q. X, Y and a conditional prevision assessment $\mathbb{P}(X|Y) = \mu$, let $G = s(X|Y - \mu) = sY(X - \mu)$ be the net random gain, where s is an arbitrary real quantity, with $s \neq 0$, and $H = (Y \neq 0)$. The assessment $\mathbb{P}(X|Y) = \mu$ is coherent if and only if: inf $G|H \cdot$ sup $G|H \leq 0$, for every s.
(without loss of generality, we can set $s = 1$)

Remark. If Y is the indicator $|H|$ of an event H, then $X|Y = X|(|H|)$ and $(Y \neq 0) \equiv (H$ *true*); then, the coherence of the assessment $\mathbb{P}(X|Y) = \mu$ reduces to the notion of coherence for the assessment $\mathbb{P}(X|H) = \mu$.

Example. We continue the study of the critical example, by examining the coherence of a given assessment $\mathbb{P}(X|Y) = \mu$. We recall that

$$(X, Y) \in \{(0, -1), (0, 1), (1, -1), (1, 1)\}\,;$$

moreover

$$H = (Y \neq 0) = \Omega\,, \quad G|H = G = Y(X - \mu)\,.$$

The values of $G|H$ associated with the values of (X, Y) are respectively:

$$g_1 = \mu,\;\; g_2 = -\mu,\;\; g_3 = -1 + \mu,\;\; g_4 = 1 - \mu;$$

hence: $\inf G|H \cdot \sup G|H \leq 0\,,\;\; \forall\, \mu.$

Another example

$$(X, Y) \in \{(0, -1), (1, 1)\}\,, \;\; \mathbb{P}(X|Y) = \mu\,.$$

We have

$$H = (Y \neq 0) = \Omega\,, \;\; G|H = G = Y(X - \mu)\,;$$

the values of $G|H$ are: $g_1 = \mu\,, \;\; g_2 = 1 - \mu\,;$

then

$$\inf G|H \cdot \sup G|H \leq 0 \iff \mu \notin (0, 1)\,;$$

i. e., if and only if: $\mu \in (-\infty, 0] \cup [1, +\infty)$.

With each μ it is associated a probability distribution on (X, Y), say $(p, 1 - p)\,, \; 0 \leq p \leq 1$, where

$$p = P(X = 0, Y = -1) = 1 - P(X = 1, Y = 1)\,.$$

By requiring that the prevision of the random gain be 0, i.e. $p\mu + (1 - p)(1 - \mu) = 0$, one has

$$p = f(\mu) = \frac{1 - \mu}{1 - 2\mu}\,,$$

with

$$\frac{1}{2} < p \leq 1\,, \quad \text{if} \;\; \mu \leq 0\,;$$

$$0 \leq p \leq \frac{1}{2}\,, \quad \text{if} \;\; \mu \geq 1\,.$$

Notice that $\mu = f^{-1}(p) = \frac{1-p}{1-2p}$; i.e., $f^{-1} = f$.

As shown by this example, the set of **coherent** *assessments* μ *may be* **not convex.**

A strong generalized compound prevision theorem

We recall that $H = (Y \neq 0)$, $\mu = \mathbb{P}(X|Y)$.
We assume that μ, $\mathbb{P}(Y|H)$, and $\mathbb{P}(XY|H)$ are finite; then, we remark that

(i) we pay μ and we receive $X|Y$, under the hypothesis H true; then, *operatively* μ is the prevision of $X|Y$, *conditional on* H;

(ii) hence, a more appropriate representation of $X|Y$ is given by:

$$X|Y = [\mu + Y(X - \mu)]|H\,;$$

(iii) then, by computing the prevision on both sides, we have $\mu = \mu + \mathbb{P}[(XY - \mu Y)|H]$ and by linearity of prevision it follows

$$\mathbb{P}(XY|H) = \mathbb{P}(X|Y)\mathbb{P}(Y|H)\,. \qquad (1)$$

Remark. If Y is a finite discrete r. q., with $Y \geq 0$, or $Y \leq 0$, it is $\mathbb{P}(Y|H) \neq 0$; then, by (1) it follows $\mathbb{P}(X|Y) = \frac{\mathbb{P}(XY|H)}{\mathbb{P}(Y|H)}$.

Notice that, as $H^c = (Y = 0)$, it follows

$$\mathbb{P}(Y|H^c) = \mathbb{P}(XY|H^c) = 0\,;$$

hence,

$$\mathbb{P}(Y) = \mathbb{P}(Y|H)P(H) + \mathbb{P}(Y|H^c)P(H^c) =$$

$$= \mathbb{P}(Y|H)P(H) = \mathbb{P}(YH)\,, \tag{2}$$

$$\mathbb{P}(XY) = \mathbb{P}(XY|H)P(H) + \mathbb{P}(XY|H^c)P(H^c) =$$

$$= \mathbb{P}(XY|H)P(H) = \mathbb{P}(XYH)\,. \tag{3}$$

Then, by (1), (2), and (3), one has

$$\mathbb{P}(XY) = \mathbb{P}(XY|H)P(H) =$$

$$= \mathbb{P}(X|Y)\mathbb{P}(Y|H)P(H) = \mathbb{P}(X|Y)\mathbb{P}(Y)\,;$$

(the formula of Lad & Dickey, which we call *weak* generalized compound prevision theorem).

The case $Y \geq 0$, or $Y \leq 0$

Let $\mathcal{C}_X, \mathcal{C}_Y$ and $\mathcal{C}$ be, respectively, the finite sets of possible values of X, Y and (X, Y).

$$X^0 = \{x_h \in \mathcal{C}_X \;:\; \exists\, (x_h, y_k) \in \mathcal{C} \;:\; y_k \neq 0\}$$

$$x_0 = \min X^0 \,, \quad x^0 = \max X^0 \,.$$

Theorem 1 Given two finite r. q. X, Y, with $Y \geq 0$ or $Y \leq 0$, the prevision assessment $\mathbb{P}(X|Y) = \mu$ is coherent iff $x_0 \leq \mu \leq x^0$.

Example.

$$(X, Y) \in \mathcal{C} = \{(0,1), (1,0), (1,1), (2,2)\} \,,$$

$\Pi =$ set of coherent assessments $\mathbb{P}(X|Y) = \mu$ on $X|Y$.
One has

$X^0 = X \,,\; x_0 = \min \mathcal{C}_X = 0 \,,\; x^0 = \max \mathcal{C}_X = 2 \,;$

the values of $G|H$, where $H = (Y \neq 0)$, are

$$g_1 = -\mu\,,\;\; g_2 = 1 - \mu\,,\;\; g_3 = 2(2 - \mu)\,;$$

such values are *all positive* (resp., *all negative*) when $\mu < 0$ (resp., $\mu > 2$);

hence every $\mu \notin [0, 2]$ is *not coherent*.

Finally, when $\mu \in [0, 2]$ one has $-\mu(2 - \mu) \leq 0$ and the condition $\inf G|H \cdot \sup G|H \leq 0$ holds.

Hence, $\Pi = [x_0, x^0] = [0, 2]$.

The case $\min Y < 0 < \max Y$.

$$X^- = \{x_h \in \mathcal{C}_X : \exists (x_h, y_k) \in \mathcal{C}, y_k < 0\}\,,$$

$$X^+ = \{x_h \in \mathcal{C}_X : \exists (x_h, y_k) \in \mathcal{C}, y_k > 0\}\,;$$

$$\mu_0 = \min\left(\max X^-, \max X^+\right),$$

$$\mu^0 = \max\left(\min X^-, \min X^+\right),$$

$$\text{if } \ \mu_0 < \mu^0\,, \ \text{ we set } \ I = (\mu_0, \mu^0)\,;$$

Moreover, we set

$$X^- < X^+\,, \ \text{ if } \ \max X^- < \min X^+\,;$$

$$X^- > X^+\,, \ \text{ if } \ \min X^- > \max X^+\,;$$

$$X^- \nsim X^+\,, \ \text{ otherwise}\,.$$

Then, we obtain

1. $X^+ < X^- \Leftrightarrow I \neq \emptyset$ and $I = (\mu_0, \mu^0)$, with $\mu_0 = \max X^+$, $\mu^0 = \min X^-$.

2. $X^+ > X^- \Leftrightarrow I \neq \emptyset$ and $I = (\mu_0, \mu^0)$, with $\mu_0 = \max X^-$, $\mu^0 = \min X^+$.

3. $X^- \nsim X^+ \Leftrightarrow I = \emptyset$.

We have

Theorem 2 Let be given two r. q. X, Y, with $\min Y < 0 < \max Y$.
If case 1, or case 2, holds, then $X^- \cap X^+ = \emptyset$ and the assessment $\mathbb{P}(X|Y) = \mu$ is coherent if and only if $\mu \notin I$.
In the case 3, the assessment $\mathbb{P}(X|Y) = \mu$ is coherent for every real number μ.

Example. We determine the set Π of coherent prevision assessments $\mathbb{P}(X|Y) = \mu$ on $X|Y$, where

$$(X, Y) \in \mathcal{C} = \{(0,1), (0,2), (1,-1), (1,-2)\}.$$

We have

$$X^- = \{1\}\,, \quad X^+ = \{0\},$$

so that $X^- \cap X^+ = \emptyset$ and $X^- \nsim X^+$.
Then, $I = (0,1)$ and, by Theorem 2, $\Pi = \Re \setminus (0,1)$; that is, μ is coherent if and only if $\mu \notin (0,1)$.

The same result follows, by observing that:
(i) $G|H = G$;
(ii) given any μ, the values of G are:

$$g_1 = -\mu, \;\; g_2 = -2\mu, \;\; g_3 = -1+\mu, \;\; g_4 = -2+2\mu;$$

(iii) if $\mu \in (0,1)$, the values of G are all negative; if $\mu \notin (0,1)$, it is: $\min G < 0\,$, $\max G > 0$.

Further work

Further developments of the research concern:

(i) coherence of a conditional prevision assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on a family of n conditional random quantities $\mathcal{F}_n = \{X_1|Y_1, \ldots, X_n|Y_n\}$;

(ii) study of general properties and methods for the checking of coherence;

(iii) generalized coherence of imprecise conditional prevision assessments, for instance interval-valued assessments like $\mathcal{A}_n = ([l_1, u_1], \ldots, [l_n, u_n])$, on $\mathcal{F}_n$.

Some results concerning (i) and (ii) have been obtained in a paper which will be presented on September at WUPES 2009.

Motivations for Poster Session

Some aspects which could be deepened during Poster Session are:

- the notion of conditional prevision, $\mathbb{P}(X|Y)$, generalize that of conditional probability, $P(E|H)$, and the familiar one of conditional prevision, $\mathbb{P}(X|H)$, where the r. q. Y is an event H.

- we can discuss an *operative* approach which, given any r. q. X and any events A, B, H, K, with $P(B|AH) = y$ and $\mathbb{P}(X|HK) = \mu$, leads to the representations

$$B|AH = (AB + yA^c)|H = AB|H + yA^c|H \, ,$$

$$X|HK = (XH + \mu H^c)|K = XH|K + \mu H^c|K \, ,$$

and, by linearity of prevision, to the formulas

$$P(AB|H) = P(B|AH)P(A|H) \, ,$$

$$\mathbb{P}(XH|K) = \mathbb{P}(X|HK)P(H|K) \, .$$

- the notion of general conditional prevision, $\mathbb{P}(X|Y)$, was introduced by Lad & Dickey, in the setting of the operational subjective theory of coherent previsions, to solve decision problems involving "state dependent preferences";

- in particular, it was applied to a "currency exchange problem" suggested by Jay Kadane.