The Pari-Mutuel Model

Renato Pelessoni[1] Paolo Vicig[1] Marco Zaffalon[2]

[1] *Department of Applied Mathematics 'B. de Finetti'*
University of Trieste, Italy

[2] *IDSIA*
Lugano, Switzerland

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The Pari-Mutuel Model (PMM)
Why study it?

- Simple, but incoherent method to achieve an upper probability $\boxed{\overline{P}_0(\cdot) = (1+\delta)P(\cdot)}$ ($\delta > 0$: *loading*).
- The PMM is a *coherent* correction of this method: $\boxed{\overline{P}(\cdot) = \min\{(1+\delta)P(\cdot), 1\}}$, or also $\boxed{\overline{P}(\cdot) = \min\{\frac{P(\cdot)}{1-\tau}, 1\}}$ ($\tau = \frac{\delta}{1+\delta}$: *taxation* or *commission*).
- A *fair* game (that using P) becomes a game favourable to the organisers (using $\overline{P}_0$ or $\overline{P}$).
- These models are common in bookmaking (Pari-Mutuel betting systems), insurance (premium pricing models), etc.
- $\overline{P}$ is coherent and 2-alternating. Useful result are available (cf. Walley (1991) for the PMM, de Cooman, Troffaes and Miranda (2005) for n-alternating upper previsions on lattices).

Main Issues

a) Generalisation of Walley's results on the PMM

b) Connections between the PMM and Risk Measurement

c) Conditioning with the PMM

d) Conditions for dilation and imprecision increase

The PMM in Walley's book

- Natural extension from a *field* $\mathcal{A}$ to an $\mathcal{A}$*-measurable* gamble X

$$\overline{E}(X) = x_\tau + (1+\delta)P((X - x_\tau)^+), \tag{1}$$

  where $x_\tau = \sup\{x \in \mathbb{R} : P(X \leq x) \leq \tau\}$ (*upper quantile*) and $(X - x_\tau)^+ = \max\{X - x_\tau, 0\}$.

- An alternative expression (Note 3, Section 3.2) (one line!):

$$\overline{E}(X) = (1-\varepsilon)P(X|X > x_\tau) + \varepsilon x_\tau \tag{2}$$

  where $\varepsilon = 1 - (1+\delta)P(X > x_\tau)$.

- When X has a continuous c.d.f. $F_X(x) = P(X \leq x)$, then

$$\overline{E}(X) = P(X|X > x_\tau). \tag{3}$$

Additional results

- *We investigate when* $\overline{E}(X) \gtreqless P(X|X > x_\tau)$ *(results depend also on so-called adherent probabilities of the c.d.f.* $F_X(x)$*).*
- *We investigate when* $\overline{E}(X) = \min\{(1+\delta)P(X), \sup X\} = \overline{P}_N(X)$ *with* $X \geq 0$ *(expected value principle, when* X *is a loss)*

  $\rightarrow \overline{P}_N(X)$ *is generally incoherent with the PMM.*

Extending the PMM from lattices

S^+: lattice of events containing $\emptyset$ and Ω
$\boldsymbol{P}_u$: partition such that $S^+ \subseteq 2^{\boldsymbol{P}_u}$
$\mathcal{L} = \mathcal{L}(\boldsymbol{P}_u)$: set of all gambles on $\boldsymbol{P}_u$

Proposition

Define the PMM $\overline{P} : S^+ \to \mathbb{R}$. *Its natural extension on* $2^{\boldsymbol{P}_u}$ *is*

$$\overline{E}(B) = \min\{(1+\delta)\tilde{P}^*(B), 1\}, \tag{4}$$

where the upper probability $\tilde{P}^*(B) = \inf\{P(A) : A \in S^+, B \Rightarrow A\}$ *is the outer (set) function of* P.

Remark

(4) *defines a coherent, imprecise PMM. Starting from a lattice introduces the imprecise PMM.*

Proposition

Define the PMM $\overline{P} : S^+ \to \mathbb{R}$.

- *Its natural extension on $\mathcal{L}(\boldsymbol{P}_u)$ is*

$$\boxed{\overline{E}(X) = x_\tau^u + (1+\delta)\overline{E}_P((X - x_\tau^u)^+)} \tag{5}$$

where $\overline{E}_P$: *natural extension of P on $\mathcal{L}$,*
x_τ^u : *(upper) quantile relative to $\tilde{P}^*$.*

- *If $(X > x_\tau^u) \neq \emptyset$,*

$$\boxed{\overline{E}(X) \leq \varepsilon^u x_\tau^u + (1 - \varepsilon^u)\overline{E}_P(X | X > x_\tau^u)} \tag{6}$$

where $\varepsilon^u \stackrel{\text{def}}{=} 1 - (1+\delta)\overline{E}_P(X > x_\tau^u)$.

Connections with Risk Measurement

$$\boxed{\overline{E}(X) = x_\tau + (1+\delta)P((X - x_\tau)^+)} \qquad (7)$$

Risk measures may be interpreted as upper previsions

- $$x_\tau \leftrightarrow VaR_\tau(X) \text{ (}\textit{Value-at-Risk}\text{ of } X \text{ at level } \tau\text{)}$$
  $$P((X - x_\tau)^+) \leftrightarrow ES_\tau(X) \text{ (}\textit{Expected Shortfall}\text{, } P \text{ expectation)}$$
  $$TVaR_\tau(X) = VaR_\tau(X) + (1+\delta)ES_\tau(X) \text{ (Denuit et al., 2005)}$$

$$\boxed{\rightarrow \overline{E}(X) \text{ is the } TVaR_\tau(X) \text{ (}\textit{Tail-Value-at-Risk}\text{)}} \qquad (8)$$

- $$P(X|X > x_\tau) \leftrightarrow CTE_\tau \text{ (}\textit{Conditional Tail Expectation}\text{)}$$

$$\boxed{\rightarrow TVaR_\tau(X) = (1-\varepsilon)CTE_\tau(X) + \varepsilon VaR_\tau(X).} \qquad (9)$$

- $TVaR_\tau$ is the natural extension on $\mathcal{L}(\boldsymbol{P}_u)$ of the PMM defined on $2^{\boldsymbol{P}_u}$. It is its *only* 2-alternating (or comonotone additive) coherent extension.
- Starting point in risk literature: a set of random variables, often a linear space with a σ-additive probability measure; expectations instead of previsions are used.
- If the PMM is defined on a lattice, we get a *new*, *coherent* risk measure, Imprecise $ITVaR_\tau$:

$$ITVaR_\tau(X) = x_\tau^u + (1+\delta)\overline{E}_P((X - x_\tau^u)^+)$$

General remark

Risk measures are functions of precise uncertainty measures in the literature, but this is not necessary nor always convenient, as shown by the PMM (and by generalisations of the Dutch Risk Measures, cf. Baroni, Pelessoni and Vicig (2009)).

Conditioning

Given the PMM on a field $\mathcal{A}$, $B \in \mathcal{A} - \{\emptyset\}$, and the conjugate $\underline{P}(A) = 1 - \overline{P}(A^c) = \max\{\frac{P(A)-\tau}{1-\tau}, 0\}$,

- when $\underline{P}(B) > 0$

$$\overline{E}(A|B) = \frac{\overline{P}(A \wedge B)}{\overline{P}(A \wedge B) + \underline{P}(A^c \wedge B)},$$

$$\underline{E}(A|B) = \frac{\underline{P}(A \wedge B)}{\underline{P}(A \wedge B) + \overline{P}(A^c \wedge B)};$$

- when $\underline{P}(B) = 0$

$\overline{E}(A|B) = 1$ (unless $A \wedge B = \emptyset$), $\underline{E}(A|B) = 0$ (unless $A \wedge B = B$).

Conditions for dilation and imprecision increase

- Given a partition $\boldsymbol{P}$, *(weak) dilation* occurs when

$$\underline{P}(A|B) \leq \underline{P}(A) \leq \overline{P}(A) \leq \overline{P}(A|B), \forall B \in \boldsymbol{P},$$

while there is *imprecision increase* when

$$\overline{P}(A) - \underline{P}(A) \leq \overline{P}(A|B) - \underline{P}(A|B), \forall B \in \boldsymbol{P}.$$

- We consider $\boldsymbol{P} = \{B, B^c\}$ (generalisations to arbitrary partitions are possible).
- Results on dilation/imprecision increase depend on the ordering of τ, $P(A \wedge B)$, $P(A \wedge B^c)$, $P(A^c \wedge B)$, $P(A^c \wedge B^c)$, with various subcases.

- Case 1: $\tau < \min\{P(A' \wedge B')\}$ (low commission case)
  - dilation occurs iff

$$\tau \geq \max \left\{ \begin{array}{c} \frac{P(A \wedge B) - P(A)P(B)}{P(A^c \wedge B^c)}, \frac{P(A)P(B) - P(A \wedge B)}{P(A \wedge B^c)}, \\ \frac{P(A \wedge B^c) - P(A)P(B^c)}{P(A^c \wedge B)}, \frac{P(A)P(B^c) - P(A \wedge B^c)}{P(A \wedge B)} \end{array} \right\}$$

  - imprecision increase always occurs!
- Case 2: $P(A) \leq \tau < \min\{P(A^c \wedge B')\}$ (A is a rare event)
  - there is dilation iff there is imprecision increase (details in the paper).

- How to circumscribe dilation/imprecision increase (assuming they should)
  - Use a coherent extension other than the natural extension. This shrinks imprecision.
  - Use a finer partition. This introduces more constraints for dilation, but... the effect on imprecision increase is uncertain.
- Imprecision increase under finer conditioning

Proposition

Let $\overline{P}$, $\underline{P}$ be conjugate and coherent, not necessarily PMM, on $\mathcal{D} \supset \{A|B, A|B_1\}$, and $A \implies B_1 \implies B$.
Then $\overline{P}(A|B_1) \geq \overline{P}(A|B)$, $\underline{P}(A|B_1) \geq \underline{P}(A|B)$.

$\implies$ Conditioning in a narrower environment may increase imprecision of the upper probability.
Thus $\overline{P}(A|B_1) - \underline{P}(A|B_1) \gtreqless \overline{P}(A|B) - \underline{P}(A|B)$.

Conclusions

- The PMM can be extended beyond Walley's framework.
- The PMM has unexpected connections with risk measurement.
- In the conditional case, dilation or imprecision increase cannot always be escaped, not even in simple situations.