Imprecise Markov Chains with Absorption

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The Markov Chain

We consider extensions to imprecision of Markov chains of the following form. Markov chain $\mathcal{X} = \{X(n), n = 0, 1, \ldots\}$ has state space $S = \{-1\} \cup C$, where $C = \{0, 1, \ldots, s\}$ is a single communicating class with all states aperiodic. These conditions are enough to ensure that the chain has a unique *limiting conditional distribution* (LCD). The generalisation to the LCD in the imprecise case was the main focus of our research.

Uses for the LCD

Situations in which the LCD is of interest

- Stability of populations;
- Catalytic reactions;
- Terminal disease progression.

The Transition Matrix

Define $s+2$ closed sets of probability distributions, $\mathcal{R}^{(i)}$, $i=-1,0,\ldots,s$. A transition matrix for the chain takes the form

$$P = \begin{pmatrix} \mathbf{r}^{(-1)} \\ \mathbf{r}^{(0)} \\ \ldots \\ \mathbf{r}^{(s)} \end{pmatrix}$$

where $\mathbf{r}^{(i)} \in \mathcal{R}^{(i)}$, for all $i \in S$. By forcing $\mathcal{R}^{(-1)} = \{(1,0,\ldots,0)\}$ we have that -1 is an absorbing state.

The Transition Matrix 2

We thus define the set of possible transition matrices:

$$\mathcal{M}(P) := \{P : \mathbf{r}^{(i)} \in \mathcal{R}^{(i)},\ \forall i \in S\}.$$

We also define

$$\mathcal{M}(P)^C := \{Q : \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{p} & Q \end{pmatrix} \in \mathcal{M}(P)\}.$$

Consideration of Time

If we assume the same matrix element of $\mathcal{M}(P)$ describes the chain at all time steps, we can consider ourselves in the time-homogeneous case. If we can change the element of $\mathcal{M}(P)$ describing the chain from time step to time step, we can consider ourselves in the time-inhomogeneous case. In both cases it can be proved that absorption is certain as time approaches infinity. This justifies our attempt to generalise the LCD to the imprecise case.

Sets of Distributions

Let $\mathcal{M}_0^C$ denote all possible distributions over C. We denote by $g(\cdot)$ the function $g(\mathbf{v}) = \frac{\mathbf{v}}{\sum_{i=0}^{s} v_i}$. In the time-homogeneous case

$$\tilde{\mathcal{M}}_n^C = \bigcup_{Q \in \mathcal{M}(P)^C} \left\{ g(\mathbf{v}Q^n) : \mathbf{v} \in \mathcal{M}_0^C, Q \in \mathcal{M}(P)^C \right\}. \tag{1}$$

In the time-inhomogeneous case we have

$$\mathcal{M}_n^C = \left\{ g(\mathbf{v}Q) : \mathbf{v} \in \mathcal{M}_{n-1}^C, Q \in \mathcal{M}(P)^C \right\}. \tag{2}$$

Sets of Distributions 2

It can be proved that $\tilde{\mathcal{M}}_n^C \subseteq \tilde{\mathcal{M}}_{n-1}^C$ and $\mathcal{M}_n^C \subseteq \mathcal{M}_{n-1}^C$ for all n. Thus, as time approaches infinity, the possible behaviour of the chain, conditioned upon non-absorption, converges to some set of distributions.

Long-Term Behaviour

We can thus define

$$\tilde{\mathcal{M}}_\infty^C = \bigcap_{n=0}^{\infty} \tilde{\mathcal{M}}_n^C$$

and

$$\mathcal{M}_\infty^C = \bigcap_{n=0}^{\infty} \mathcal{M}_n^C.$$

Long-Term Behaviour

In the time-homogeneous case, $\tilde{\mathcal{M}}_{\infty}^{C}$ is simply the set of LCDs for the elements of $\mathcal{M}(P)$. Further, it can be shown that

$$\mathcal{M}_{\infty}^{C} = \{g(\mathbf{v}Q) : \mathbf{v} \in \mathcal{M}_{\infty}^{C}, Q \in \mathcal{M}(P)^{C}\}$$

making the above set the generalisation to the LCD in the time-inhomogeneous case. It can be proved that this sets is reached as time approaches infinity irrespective of the set of initial distributions chosen.

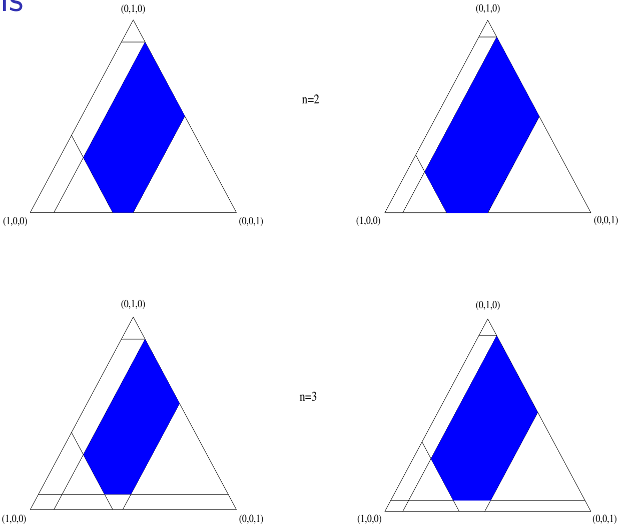
The Chain

Consider a Markov chain with state space $S = \{-1\} \cup C$ where $C = \{0, 1, 2\}$. Let each $P \in \mathcal{M}(P)$ takes the form

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 0 & 1-a & 0 \\ 0 & b & 0 & 1-b \\ 0 & 0 & c & 1-c \end{pmatrix}$$

where $a \in [0.1, 0.3]$, $b \in [0.5, 0.6]$, and $c \in [0.67, 0.73]$. We compare the effects of considering this chain in the time-homogeneous case and the time-inhomogeneous case.

Diagrams



Diagrams