Natural extension as a limit of regular extensions

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Goal of the paper

Given a number of conditional lower previsions satisfying either:

- Weak coherence;
- Coherence,

we study the smallest extension to a bigger domain that is consistent with them.

Our work is based on earlier results by Walley, Pelessoni and Vicig.

Outline

1. Conditional lower previsions.

2. Extension of weakly coherent assessments.

3. Extension of coherent assessments.

4. Conclusions and open problems.

Conditional lower previsions

Consider variables $\{X_1, \ldots, X_n\}$, taking values in *finite* spaces $\mathscr{X}_1, \ldots, \mathscr{X}_n$.

Given disjoint $O, I \subseteq \{1, \ldots, n\}$, the conditional lower prevision $\underline{P}(X_O|X_I)$ represents the information that the variables in I provide about the variables in O.

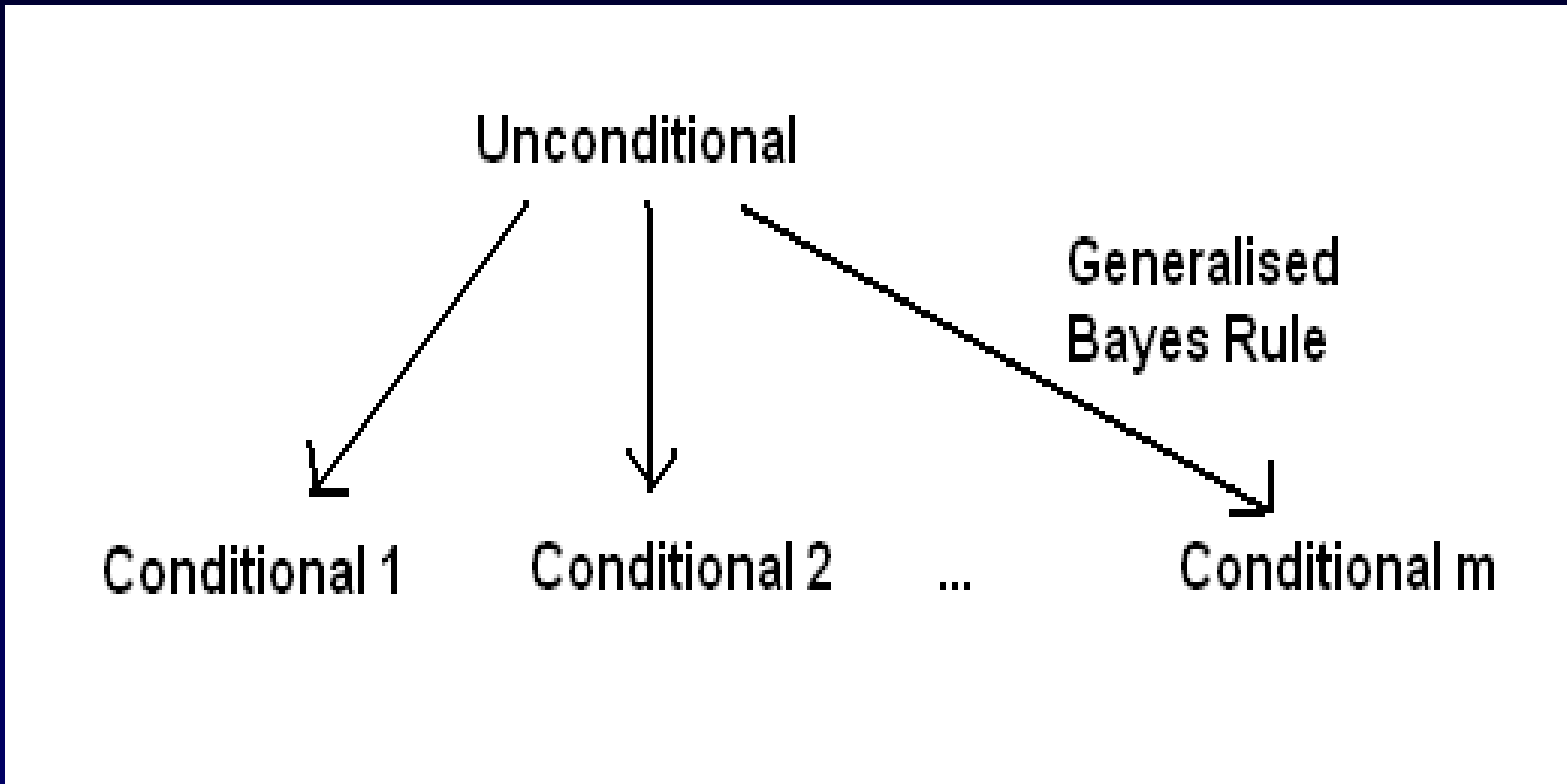
We interpret $\underline{P}(f|x)$ as the supremum acceptable buying price for a gamble f if we learn that X_I has taken the value x.

The problem

We consider a number of conditional lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$, we want to define a new conditional $\underline{P}_{m+1}(X_{O_{m+1}}|X_{I_{m+1}})$ which is *compatible* with them. We use two procedures:

- Natural extension, where we use the behavioural implications of the assessments already made.
- If we have a set of compatible unconditional previsions, their regular extension consists in applying Bayes' rule whenever possible and then take envelopes.

Weak coherence



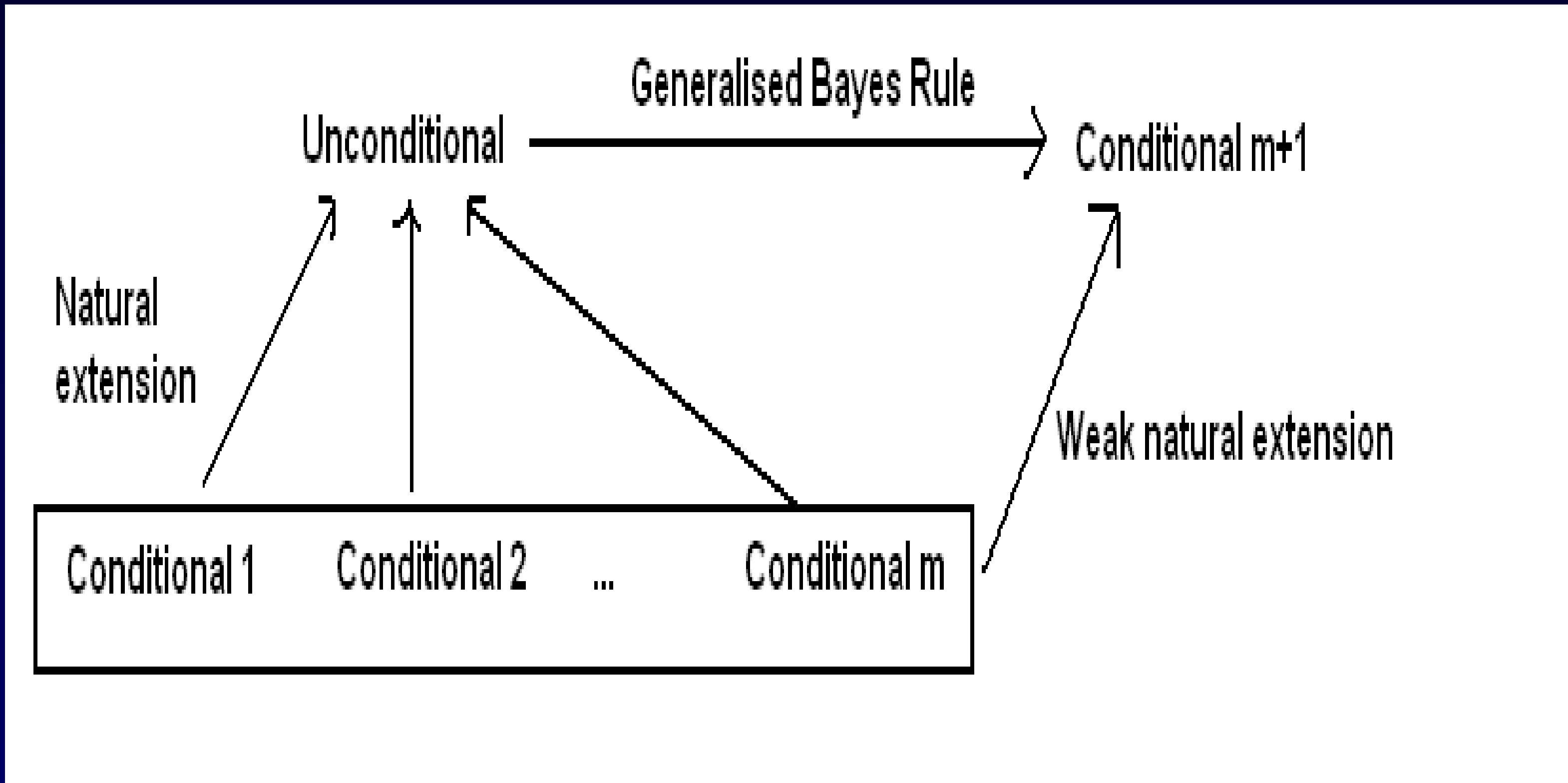
- This means that a supremum acceptable conditional buying prices cannot be increased taken other acceptable buying prices.

Smallest compatible joint

- The smallest coherent lower prevision which is weakly coherent with $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ can be obtained with the procedure of natural extension.

- From the behavioural point of view, this condition is too weak, because it does not detect inconsistencies on sets of probability zero.

Weakly coherent natural extension



$$\underline{P}_{m+1}(f|z_{m+1}) = \begin{cases} \min_{x \in \pi^{-1}_{I_{m+1}}(z_{m+1})} f(x) & \text{if } \underline{P}(z_{m+1}) = 0 \\ \min\{P(f|z_{m+1}) : P \geq \underline{P}\} & \text{otherwise.} \end{cases}$$

Coherence

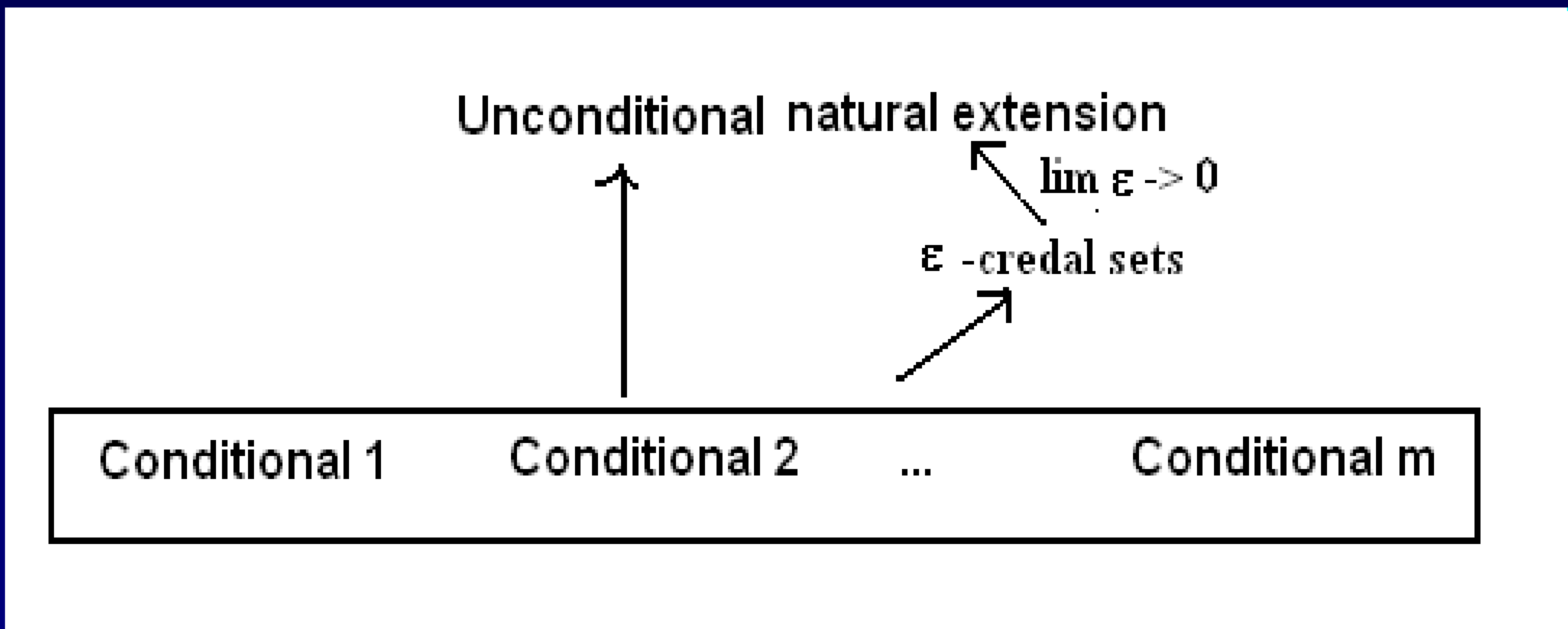
- The stronger notion of coherence deals with the problem of sets of zero probability by looking only at the sets where the acceptable transactions are non-trivial.
- Given a number of coherent conditional lower previsions, their natural extension provides their behavioural consequences on other gambles. It is the smallest (conditional) lower prevision which is coherent with them.
- Our goal is to give an easy characterisation of the natural extension.

ε-approximations

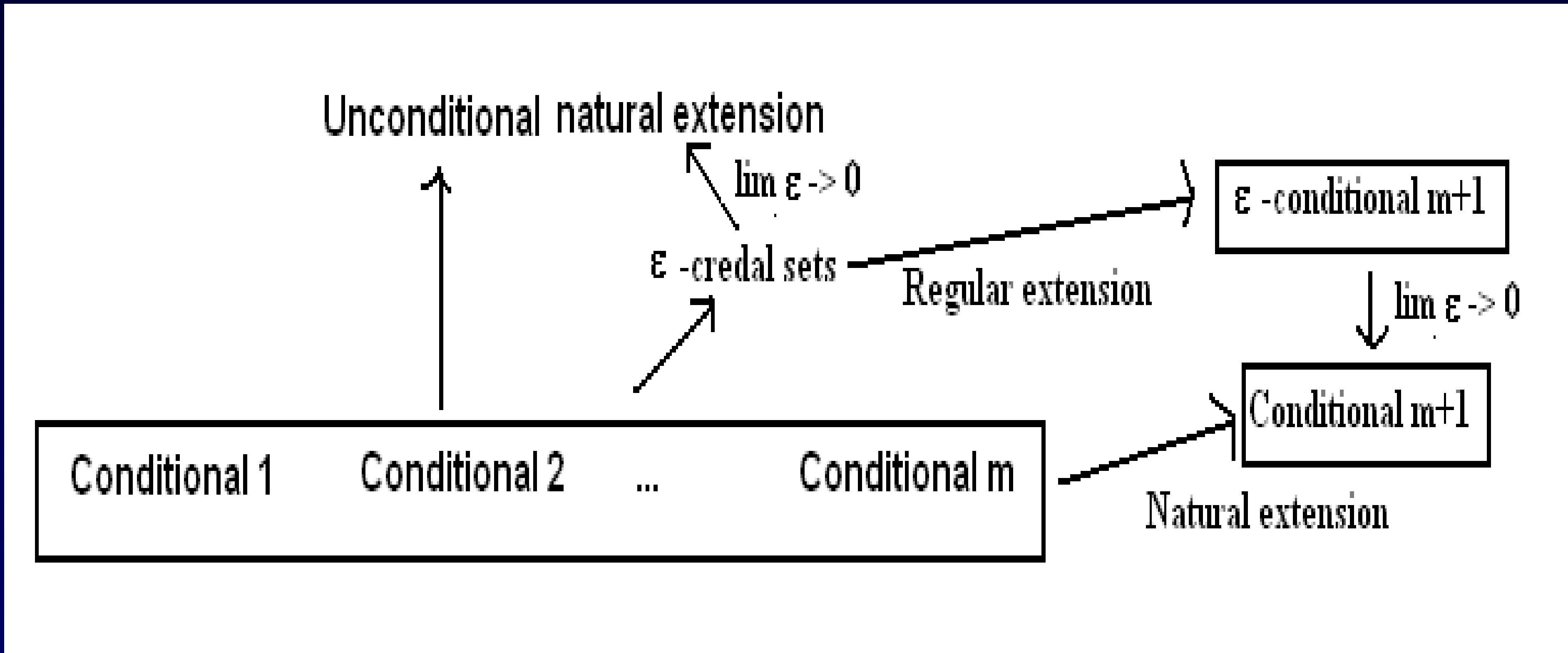
Given $\varepsilon \geq 0$, let $\mathscr{M}(\varepsilon)$ be the set of linear previsions s.t.

$$P(f_j \pi_{I_j}^{-1}(z_j)) \geq P(z_j)(\underline{P}_j(f_j|z_j) - \varepsilon R(f_j)),$$

where $R(f_j) = \max f_j - \min f_j$.

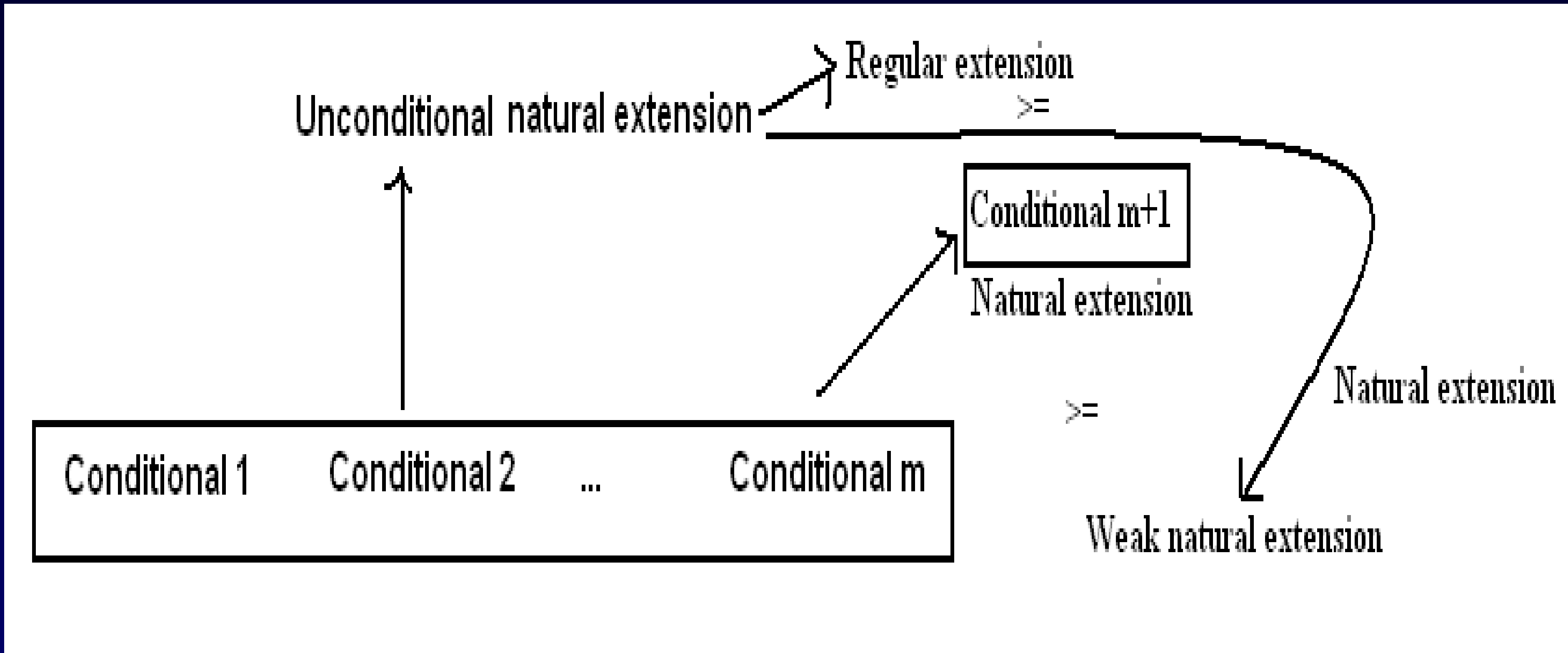


Main result



Unconditional natural extension
lim ε -> 0
ε -credal sets
ε -conditional m+1
Regular extension
lim ε -> 0
Conditional m+1
Conditional 1
Conditional 2
...
Conditional m
Natural extension

Do we really need ε?



- If $\underline{P}(z_{m+1}) > 0$, then the natural extension coincides with the weak natural extension.

Conclusions

- Coherent conditional lower previsions are always the limit of conditional lower previsions defined by regular extension.
- The smallest weakly coherent extension does not always coincide with the natural extension.
- The difference is caused by conditioning on sets of zero lower probability.

Open problems

- Extension to infinite spaces.
- Inclusion of structural judgements.
- Comparison with the zero-layers.