ISIPTA'09 July 14th, 2009

# A Minimum Distance Estimator in an Imprecise Probability Model

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## Computational Aspects and Applications

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Department of Mathematics, University of Bayreuth (Germany)

- Diploma Thesis in Robust Asymptotic Statistics (Helmut Rieder, University of Bayreuth)
- Ph.D. Thesis: "Data-Based Decisions under Complex Uncertainty" (Thomas Augustin, LMU Munich)
- Now: University of Bayreuth
  Working group: Andreas Christmann (Machine Learning) and Helmut Rieder (Robust Statistics)
- Research Interests:
  - Mathematical Foundations of Imprecise Probabilities
  - (Statistical) Decision Theory under Imprecise Probabilities
  - Robust Statistics
  - Nonparametric Statistics

# Estimation: Classical Statistics and Generalizations

Data $x_1, \ldots, x_n$ are modeled by random variables

$$X_1\,, \ldots\,, X_n \;\sim\; P_0 \qquad\qquad \text{i.i.d.}$$

$P_0$ the true distribution (precise probability measure)

- classical statistics
  - $(P_\theta)_{\theta\in\Theta}$ : a parametric model (precise distributions)
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**Goal:** Estimate $\theta_0$

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**. . . may be found on the poster!**

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- Simple **idea**:
  - Construct empirical measure $\mathbb{P}_n$ from the data
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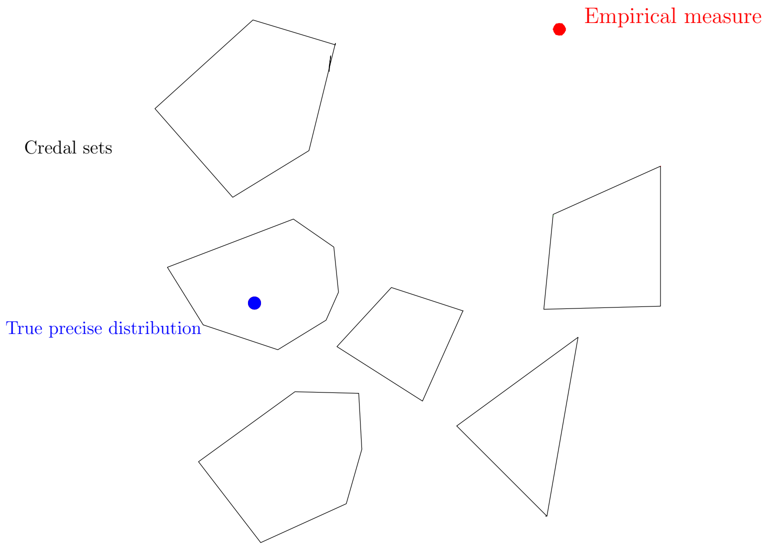
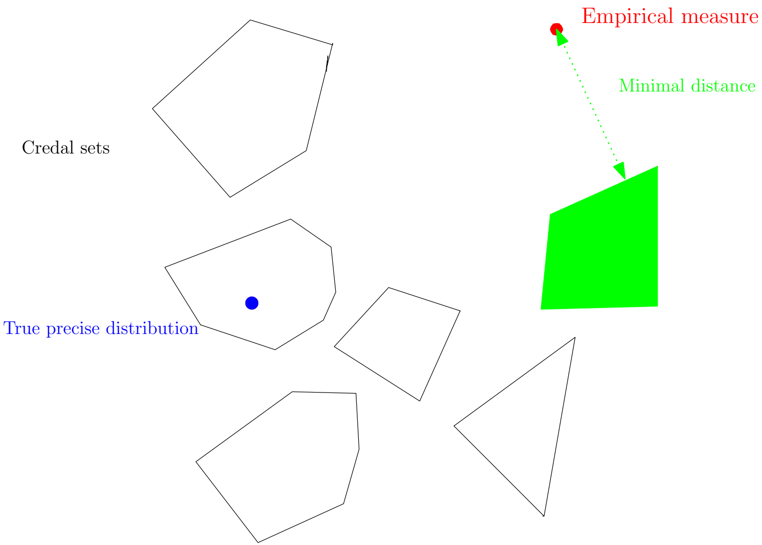
- **Definition**:

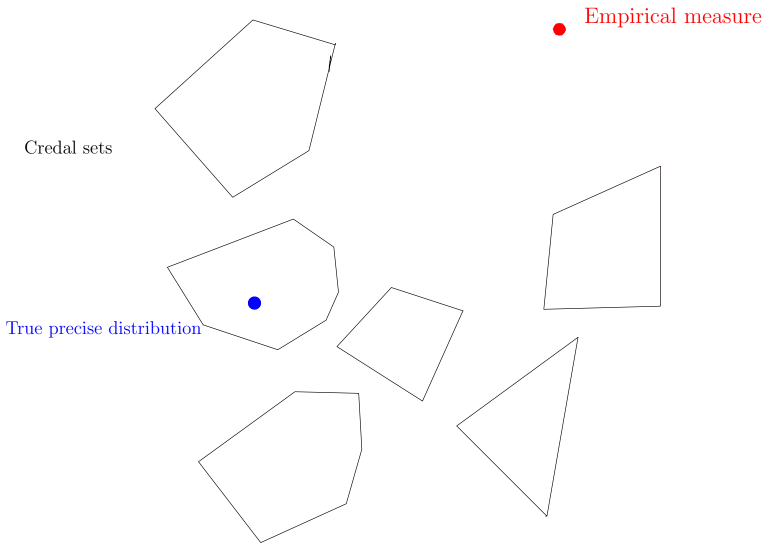
  Estimator $\hat{\theta}_n$ minimizes

$$\theta \;\mapsto\; d\big(\mathbb{P}_n, \mathcal{M}_\theta\big) \;=\; \inf_{P_\theta \in \mathcal{M}_\theta} d\big(\mathbb{P}_n, P_\theta\big)$$

  Distance $d$ : total variation


Empirical measure
Credal sets
True precise distribution


Empirical measure
Minimal distance
Credal sets
True precise distribution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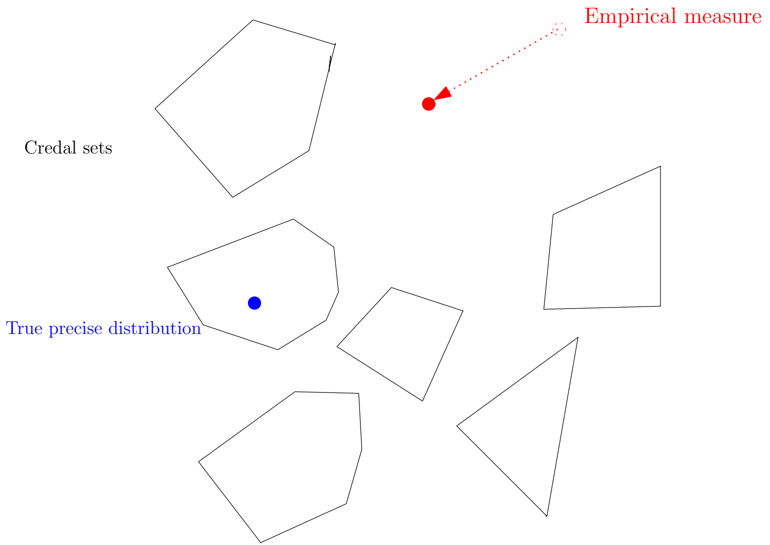
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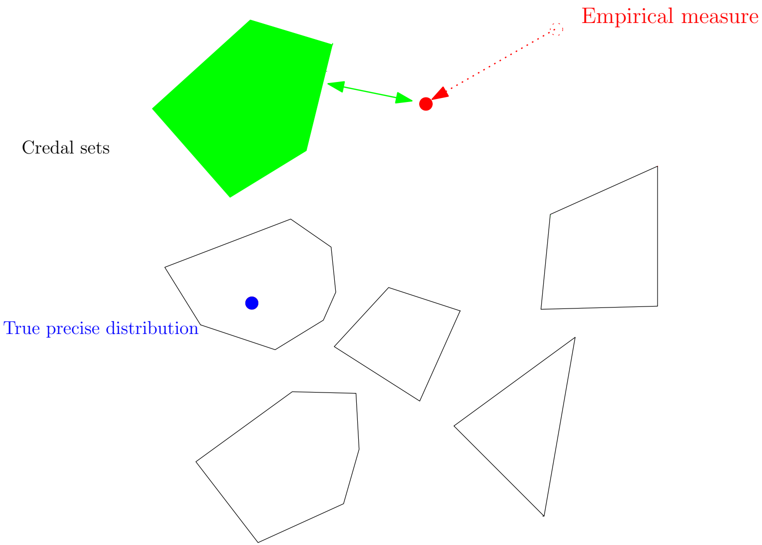
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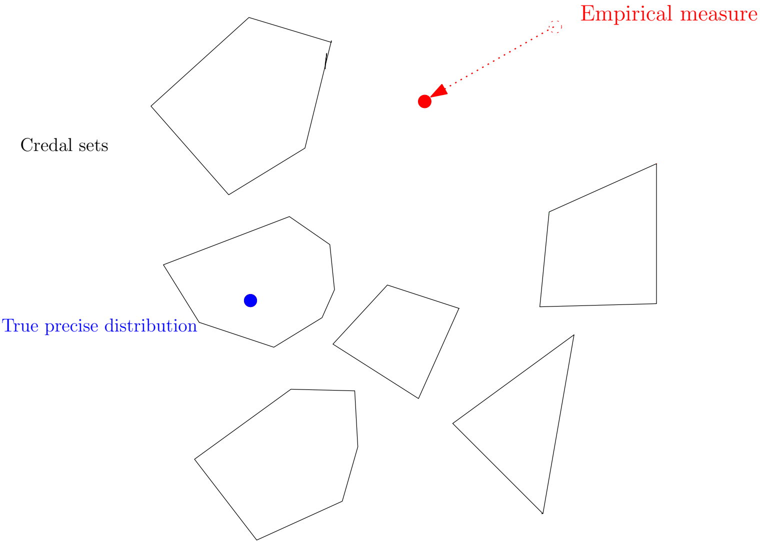
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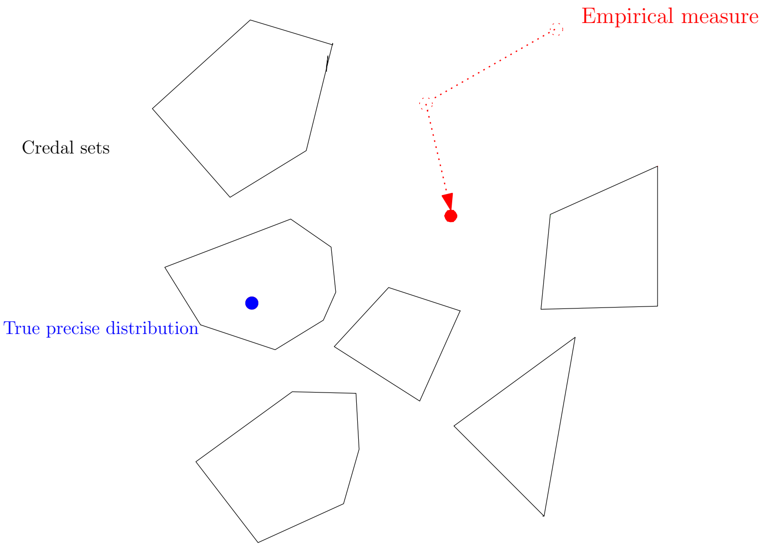
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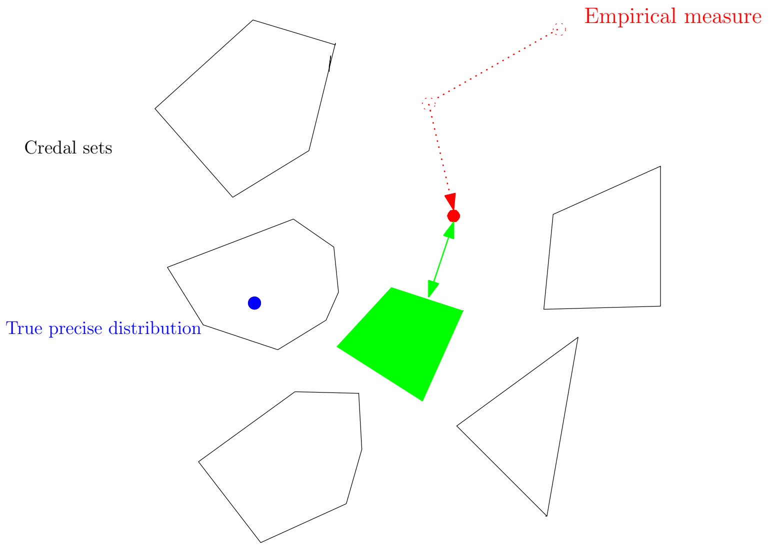
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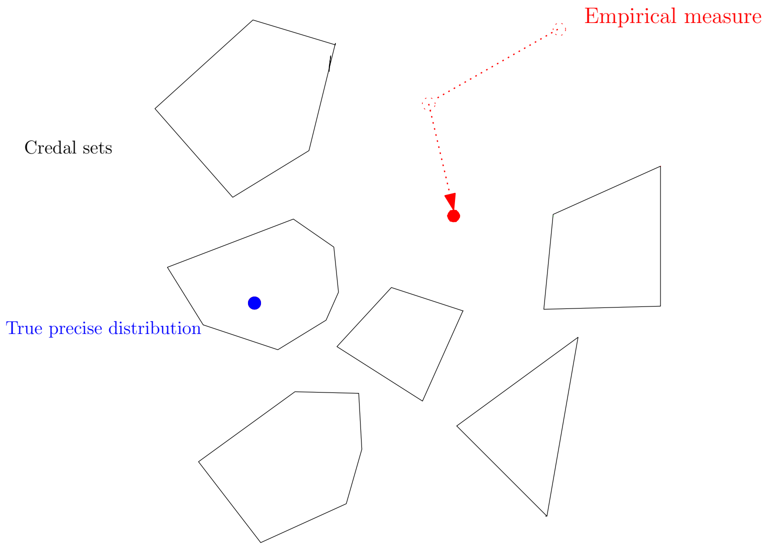
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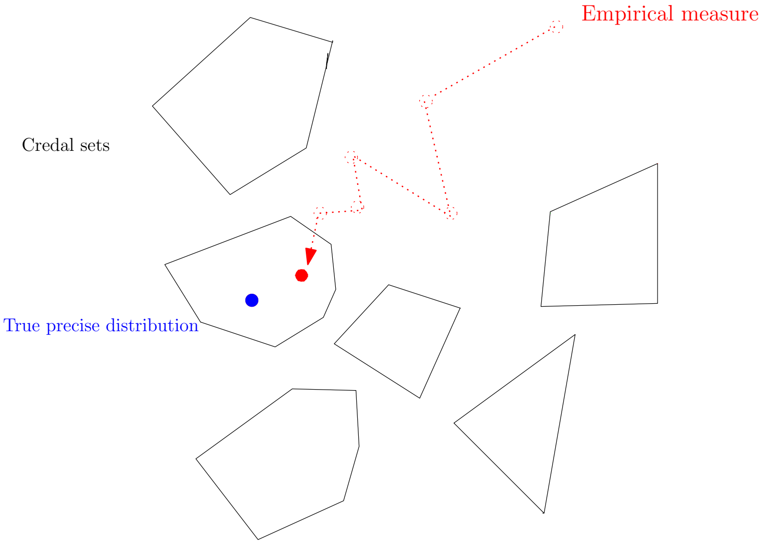
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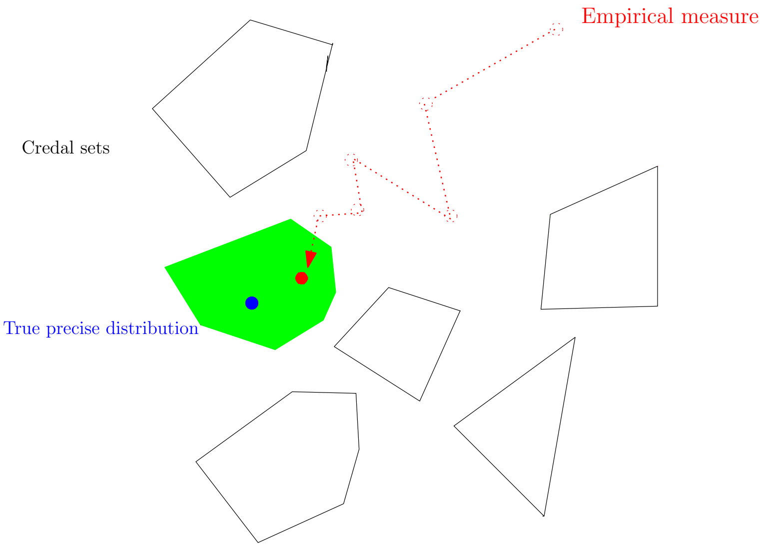
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# Calculations

- Replace original sample space $(\mathcal{X}, \mathcal{B})$ by a suitable discrete sample space $(\mathcal{X}, \mathcal{C})$
- Approximate calculation of $d\left(\mathbb{P}_n, \mathcal{M}_\theta\right)$ by linear programming

# Calculations

- Replace original sample space $(\mathcal{X}, \mathcal{B})$ by a suitable discrete sample space $(\mathcal{X}, \mathcal{C})$
- Approximate calculation of $d\big(\mathbb{P}_n, \mathcal{M}_\theta\big)$ by linear programming:

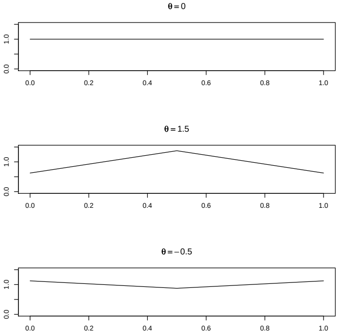
$$\sum_{j \in \mathcal{J}_1} q_j - \gamma_j \quad \longrightarrow \quad \text{max!}$$

$$\begin{aligned}
&\textstyle\sum_{j=1}^{r} q_j = 1 \\
&\textstyle\sum_{j=1}^{r} q_j h_k(c_j) \leq \overline{P}_\theta[f_k] + \varepsilon_\theta^{(k)} \ \forall\, k \in \{1, \dots, s\} \\
&q_j - \gamma_j \ \leq \ \tfrac{n_j}{n} \ \forall\, j \in \mathcal{J}_1 \\
&(q_1, \dots, q_r) \ \in \ \mathbb{R}^r, \ q_j \geq 0 \quad \forall\, j \in \{1, \dots, r\}, \\
&(\gamma_j)_{j \in \mathcal{J}_1} \ \subset \ \mathbb{R}, \ \gamma_j \geq 0 \quad \forall\, j \in \mathcal{J}_1
\end{aligned}$$

- Implemented as R-Package `imprProbEst`; see Hable (2008).

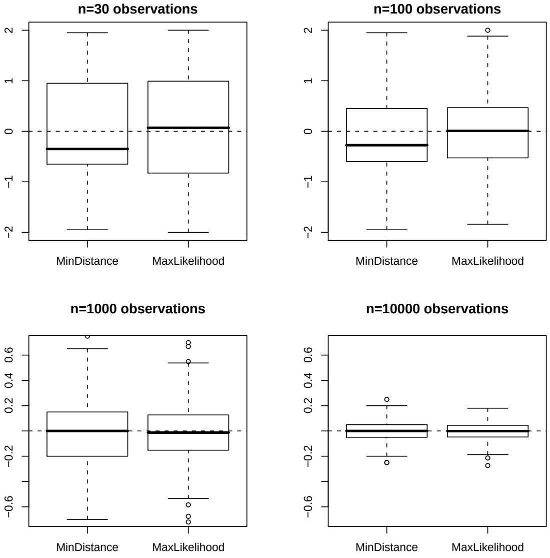
# Simulation Study: Model 1

Data from an ideal, precise model with densities such as




- **Maximum likelihood estimator:** complete information about precise model
- **Minimum distance estimator:** only imprecise information about precise model

# Simulation Study: Model 1



# Simulation Study: Model 2

Model: Approximate Poisson Distribution

- **90%** of the data (in average) from **Poisson** distribution
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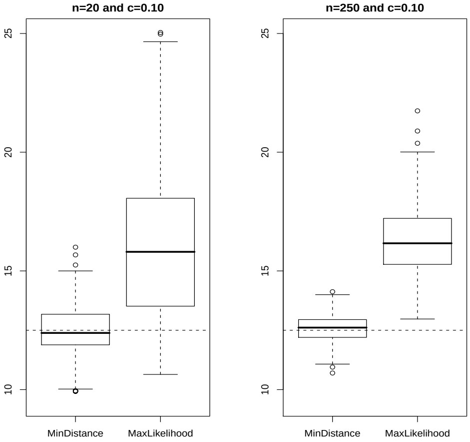

# Real Data Set: Linear Regression

NHANES: persons' body weight depending on persons' height

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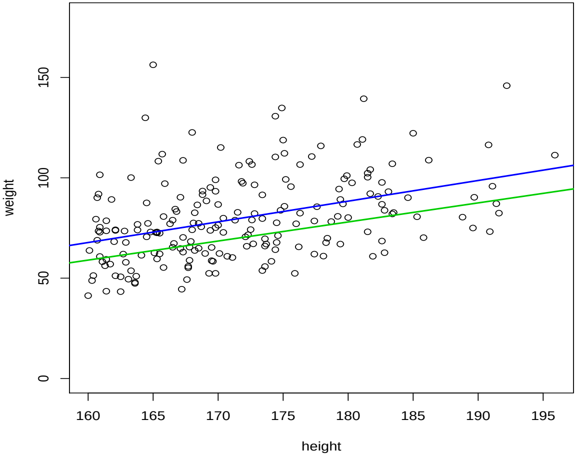


Minimum Distance Estimator Least Squares Estimator

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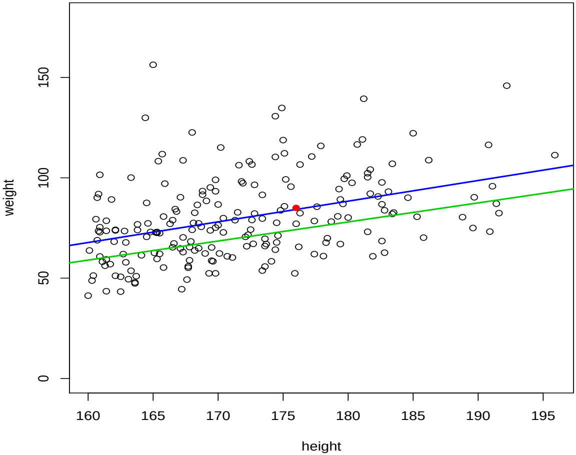


Minimum Distance Estimator     Least Squares Estimator

# References


- Hable, R. (2008). *imprProbEst: Minimum distance estimation in an imprecise probability model*. Contributed R-Package on CRAN, Version 1.0, 2008-10-23; maintainer Hable, R.
- Hable, R. (2009a). Data-based decisions under complex uncertainty. *Ph.D. Thesis*. `http://edoc.ub.uni-muenchen.de/9874/`
- Hable, R. (2009b). A minimum distance estimator in an imprecise probability model – computational aspects and applications. *6th International Symposium on Imprecise Probability: Theories and Applications (ISIPTA'09)*.


The handout to this talk is also available on my homepage

**http://www.staff.uni-bayreuth.de/˜btms04/index.html**