Sets of Desirable Gambles and Credal Sets

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Abstract

Sets of desirable gambles were proposed by Walley [7] as a general theory of imprecise probability. The main reasons for this are: it is a very general model, including as particular cases most of the existing theories for imprecise probability; it has a deep and simple axiomatic justification; and mathematical definitions are natural and intuitive. However, there is still a lot of work to be done until the theory of desirable gambles is operative for its use in general reasoning tasks. This paper gives an overview of some of the fundamental concepts expressed in terms of desirable gambles in the finite case, gives a characterization of regular extension, and studies the nature of maximally coherent sets of gambles.

Keywords. Desirable gambles, regular extension, zero probabilities, sets of probability measures.

1 Introduction

Sets of desirable gambles are a powerful and simple model for representing and reasoning with imprecise probabilities. For these reasons, they were proposed by Walley [7] as a general model for imprecise probability after studying the limitations of other models.

The axioms for desirable gambles were introduced by Williams [9] and Walley studied them in Appendix F of his book [6]. They were also considered in [5] as a basic for a logical approach to probability. They are mathematically equivalent to partial probability orderings [1, 3], but they are simpler [7]. Because of this, desirable gambles are a more suitable theory of uncertainty. Even though, their use in the literature is very scarce. In many cases, it is possible to find papers based on other representations, as for example lower and upper previsions, in which the rules for inference are deduced making arguments which are based on desirability. This makes desirability a more primitive notion.

Moral [4] recently studied the concept of epistemic irrelevance in terms of desirable gambles which resulted in a very natural approach to this notion, as it was possible to show a number of properties in a simple form.

In this paper, we give an overview of some of the main concepts of desirable gambles in the finite case, showing the difference between desirable gambles and almost desirable gambles (Section 2). Then, we study the concept of conditioning, showing how the rules of conditioning for lower previsions can be obtained from the simple definition of conditioning for sets of desirable gambles and giving an axiomatic justification of regular extension (Section 3). One of the problems associated to the use of desirable gambles is the lack of effective methods of representing information and algorithms to make inference from available information. Section 4 discusses this issue and shows that there are algorithms in the literature which can be directly applied in this theory. Finally Section 5 studies the case of maximally coherent sets of desirable gambles. These sets have always an associated precise probability measure. But, as sets of desirable gambles contain more information than probability measures, we prove that we can associate a more complex structure to the maximal coherent sets: a sequence of probability measures, each one of them defined in the set in which the previous measure in the sequence assigns a zero probability, similar to the sequences defined in [2]. We also show that a general coherent set can be expressed in terms of maximal (precise) coherent sets.

2 Sets of Desirable Gambles

Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ denote the (finite) set of outcomes. We assume that there is an unkown true value belonging to Ω. A *gamble* on Ω is a bounded mapping from Ω to $\mathbb{R}$, i.e., $X : \Omega \to \mathbb{R}$. Gambles are used to represent an agent's beliefs and information. If an agent accept a gamble X, then the value $X(\omega)$ rep-

resents the reward she would obtain if ω is the true unknown value (this value can be negative and then it represents a loss).

Let $\mathcal{L}$ denote the set of all gambles defined on Ω. For $X, Y \in \mathcal{L}$, let $X \geq Y$ mean that $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$, and let $X > Y$ mean that $X \geq Y$ and $X(\omega) > Y(\omega)$ for some $\omega \in \Omega$.

A subset $\mathcal{D}$ of $\mathcal{L}$ is said to be a *coherent set of desirable gambles* relative to $\mathcal{L}$ [7] when it satisfies the following four axioms:

D1. $0 \notin \mathcal{D}$,

D2. if $X \in \mathcal{L}$ and $X > 0$ then $X \in \mathcal{D}$,

D3. if $X \in \mathcal{D}$ and $c \in \mathbb{R}^+$ then $cX \in \mathcal{D}$,

D4. if $X \in \mathcal{D}$ and $Y \in \mathcal{D}$ then $X + Y \in \mathcal{D}$.

In what follows, $\mathcal{D}$ is assumed to be a coherent set of gambles. We assume that information is represented by means of a coherent set of gambles. These rules represent the consistency conditions for the gambles that are considered desirable. For example, Axiom D4 says that if we consider as desirable X and Y, then we should consider as desirable the gamble resulting from adding the rewards of both gambles. Axiom D2 says that a positive gamble (we can win but never lose) is always desirable.

The null gamble is neutral and then it is not included in the set of really desirable gambles, but this is not an important fact. In some cases, as in [4, 6], the null gamble has been considered desirable. The real important condition for coherence is that if $X < 0$, then $X \notin \mathcal{D}$ (avoiding *partial loss*). In our approach, this condition is a consequence of D1 and the other axioms (D2 and D4). But both options are completely equivalent, in the sense that the only difference is the inclusion of the null gamble in the set of desirable gambles and this does not have any difference in practice. The only consequence of taking one of the two possible options is that some mathematical definitions have to be changed (for example, conditioning is different if we accept the null gamble). Walley first considered the null gamble desirable in [6], but then he changed to consider it non desirable in [4]. In this moment, we also consider that this last option is simpler and more intuitive.

The *lower prevision induced by* $\mathcal{D}$ is the function $\underline{P} : \mathcal{L} \to \mathbb{R}$ defined as follows: $\underline{P}(X) = \sup\{c : X - c \in \mathcal{D}\}$.

The *upper prevision induced by* $\mathcal{D}$ is the function $\overline{P} : \mathcal{L} \to \mathbb{R}$ defined as follows: $\overline{P}(X) = \inf\{c : c - X \in \mathcal{D}\}$.

The set of linear previsions induced by $\mathcal{D}$ is defined as:

$$\mathcal{P}_{\mathcal{D}} = \{P : P(X) \geq 0 \text{ for all } X \in \mathcal{D}\}.$$

$\mathcal{P}_{\mathcal{D}}$ is always a *credal set* (a closed and convex set of probability measures). $\underline{P}$ and $\overline{P}$ are dual and they respectively coincide with the pointwise infimum and the pointwise supremum of $P \in \mathcal{P}_{\mathcal{D}}$. There can be two different sets of desirable gambles $\mathcal{D} \neq \mathcal{D}'$ inducing the same class of linear previsions $\mathcal{P}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}'}$.

Conversely, given a set of linear previsions $\mathcal{P}$, define

$$\mathcal{D}_{\mathcal{P}} = \{X \in \mathcal{L} : P(X) > 0, \forall P \in \mathcal{P}\} \cup \{X : X > 0\}.$$

$\mathcal{D}_{\mathcal{P}}$ is called the set of *strictly desirable gambles* associated to $\mathcal{P}$ [6].

$\mathcal{D}_{\mathcal{P}}$ is coherent and, if $\mathcal{P}$ has been induced by a set of desirable gambles $\mathcal{D}$, then $\mathcal{D}_{\mathcal{P}}$ is a subset of it. In other words, the following inclusion holds:

$$\mathcal{D}_{\mathcal{P}_{\mathcal{D}}} \subseteq \mathcal{D}$$

$\mathcal{D}_{\mathcal{P}}$ is the smallest set of gambles associated to a credal set $\mathcal{P}$.

$\mathcal{P}$ can be recovered from $\mathcal{D}_{\mathcal{P}}$ by

$$\mathcal{P} = \mathcal{P}_{\mathcal{D}_{\mathcal{P}}}. \tag{1}$$

Another possible set of desirable gambles associated to $\mathcal{P}$, but with more gambles in it is:

$$\begin{aligned}\mathcal{D}'_{\mathcal{P}} = \{X \in \mathcal{L} : P(X) \geq 0, \ \forall P \in \mathcal{P} \text{ and} \\ \exists P \in \mathcal{P}, \text{ with } P(X) > 0\} \cup \{X : X > 0\}\end{aligned}$$

A coherent set $\mathcal{D}$ of *almost desirable gambles* is a set of gambles which satisfies axioms D2, D3, and D4 and the following two axioms (the first one is a modification of the corresponding axiom for desirable gambles. The new version is called *avoiding sure loss*):

D1'. $-1 \notin \mathcal{D}$,

D5. if $X + \epsilon \in \mathcal{D}, \forall \epsilon > 0$, then $X \in \mathcal{D}$

A set of almost desirable gambles $\mathcal{D}$ can define a lower prevision, an upper prevision, and a credal set, by means of expressions completely analogous to the case of desirable gambles. But now, from a credal set $\mathcal{P}$, the associated set of almost desirable gambles $\mathcal{D}$ is given by:

$$\mathcal{D}^*_{\mathcal{P}} = \{X \in \mathcal{L} : P(X) \geq 0, \ \forall P \in \mathcal{P}\} \tag{2}$$

Intuitively, the set of desirable gambles contains all the gambles that are really desirable, i.e. the agent has reasons to accept them as desirable. The set of almost desirable gambles also includes all the gambles that are limit of desirable gambles, though some of them as the null gamble is not really desirable. $\mathcal{D}_{\mathcal{P}}$, the set of striclty desirable gambles associated to $\mathcal{P}$ is the interior of $\mathcal{D}^*_{\mathcal{P}}$ in the supremum norm topology [6].

If $\mathcal{D}$ is a coherent set of desirable gambles, then $\mathcal{D}^*$ will be the coherent set of almost desirable gambles obtained by adding to it the gambles resulting of the application of Axiom D5 (closure). Both $\mathcal{D}$ and $\mathcal{D}^*$ always define the same credal set. If the credal set is $\mathcal{P}$, then $\mathcal{D}_{\mathcal{P}} \subseteq \mathcal{D} \subseteq \mathcal{D}^*$. $\mathcal{D}_{\mathcal{P}}$ contains the strictly desirable gambles. If a gamble X is in $\mathcal{D}_{\mathcal{P}}$, then there is a $\delta > 0$ such that $X - \delta \in \mathcal{D}_{\mathcal{P}}$, i.e. even paying a quantity δ, the gamble continues being desirable. $\mathcal{D}^*$ contains more gambles, all the gambles such that for any $\epsilon > 0$, $X + \epsilon$ is desirable, i.e., if we receive any positive quantity, this is enough to make the gamble desirable (but the gamble alone may not be desirable). $\mathcal{D}$ is the set of gambles that are considered desirable by an agent without any additional consideration in the limit.

Coherent sets of almost desirable gambles and credal sets are equivalent, in the sense that there is a one-to-one correspondence between these two families. If $\mathcal{D}$ is a set of almost desirable gambles: $\mathcal{D}^*_{\mathcal{P}_{\mathcal{D}}} = \mathcal{D}$. A credal set is a convex and closed set of probabilities and an almost desirable gamble can be interpreted as a linear restriction on the credal set by means of expression $P(X) \geq 0$. The difference between desirable and almost desirable gambles is that a set of almost desirable gambles is always closed, and a set of desirable gambles is never closed (the null is the limit of desirable gambles and is never desirable) but not necessarily open either. The set of strictly desirable gambles is always open. Axioms can be also defined for strict desirable gambles [6] and it is possible to show the equivalence between sets of strict desirable gambles and credal sets.

Example 1 *Consider the credal set $\mathcal{P}$ represented in Figure 2 for a frame with three elements $\{\omega_1, \omega_2, \omega_3\}$, where each point is a probability mass function with values determined by the distances to the triangle edges. Imagine that $\mathcal{D}$ and $\mathcal{D}^*$ are a set of desirable gambles and the set of almost desirable gambles associated to it. A gamble can be associated to a linear restriction about the probabilities through the inequality $P(X) \geq 0$. If this inequality is not trivial in the set of probabilities (X is trivial if $X \geq 0$), then in the triangle we will see the inequality as a segment dividing the triangle in two parts and a direction determining in which of the two parts the inequality is verified. So, a non trivial gamble X can be associated with a segment and a direction. A gamble is almost desirable if all the probabilities in the credal set verify the restriction. In the figure, X_1 and X_3 are almost desirable and X_2 is not as there is a probability in $\mathcal{P}$ not verifying the inequality associated to X_2. X_1 is also strictly desirable. For desirability we have a necessary condition: if X is desirable then $P(X) \geq 0$ for any $P \in \mathcal{P}$. So, as X_2 does not verify it, it can not be desirable. We also have a sufficient condition: if $P(X) > 0$, for any P, then X has to be strictly desirable and desirable. So X_1 is desirable and strictly desirable. The difference is in those gambles X, for which $P(X) \geq 0$ for any P, but $P(X) = 0$ for some P. This gamble is almost desirable and can not be strictly desirable, but it can be desirable or not desirable. So, it is not determined whether gamble X_3 (touching the border of the credal set) is or is not desirable. These gambles in the border determine the difference between desirability, almost desirability, and strict desirability. They have behabioural consequences, in particular after conditioning to events of probability 0.*

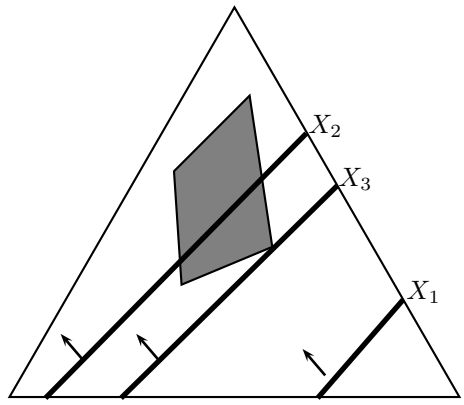

Figure 1: Desirable and almost desirable gambles

If $\mathcal{G}$ is an arbitrary set of gambles, then the set of all gambles obtained by applying axioms D2, D3, and D4 is called the *set of gambles generated* by $\mathcal{G}$ and it is denoted by $\overline{\mathcal{G}}$. If this set is coherent ($0 \notin \overline{\mathcal{G}}$) then it will be called its *natural extension* (the minimum coherent set containing $\mathcal{G}$). If $0 \in \overline{\mathcal{G}}$ we will say that $\mathcal{G}$ is incoherent. If $X < 0$ and $X \in \overline{\mathcal{G}}$ we will say that $\mathcal{G}$ does not *avoid partial loss.*

It is an immediate result that

$$\overline{\mathcal{G}} = \{\sum_{i=1}^{n} \lambda_i X_i \,:\, \lambda_i > 0,\ [X_i \in \mathcal{G} \text{ or } X_i > 0] \quad i \leq n \in \mathbb{N},\, n \geq 1\}.$$

Walley [6] considers the gambles that dominate (are greater or equal) than the positive linear combination

of gambles in $\mathcal{G}$. Our expression with equality is equivalent as we allow to combine positive gambles, except that we avoid to add the 0 gamble.

3 Conditioning

Let us consider a set of desirable gambles $\mathcal{D}$ on Ω. Let B denote (the indicator function of) an arbitrary subset of Ω. The set of *B-desirable gambles* ([6], Section 6.1.6) can be defined as follows:

$$\mathcal{D}_B = \{X \in \mathcal{L} \,:\, BX \in \mathcal{D}\} \cup \{X \,:\, X > 0\}.$$

This set will be also called the set of *conditional desirable gambles given B*. This set is determined by those gambles Y that are desirable and that outside of B are null, i.e. nothing happens if B does not occur. A gamble X belongs to $\mathcal{D}_B$ if BX is equal to one of these gambles or is positive.

The following results relate this definition with the usual concept in the associated credal set, consisting in computing the conditional probability given B of all the probability measures in the credal set (when $P(B) > 0$ for all the probabilities). In all of them, $\mathcal{D}^*$ is the set of almost desirable gambles associated to the set of desirable gambles $\mathcal{D}$.

Lemma 1 *Let $\mathcal{D} \subset \mathcal{L}$ be a coherent set of desirable gambles and B a subset of Ω such that $\underline{P}(B) > 0$. Then:*

$$X \in \mathcal{D}^* \Rightarrow X + \epsilon\, B \in \mathcal{D},\ \forall\, \epsilon > 0.$$

Proof: According to the above hypotheses, $\underline{P}(B) > 0$ and thus, there exists some $c > 0$ such that $B - c \in \mathcal{D}$. Furthermore, the gamble $X + \epsilon c$ is assumed to belong to $\mathcal{D}$, for all $\epsilon > 0$. By the coherence of $\mathcal{D}$, the gambles $\epsilon(B - c) = \epsilon B - \epsilon c$ and $X + \epsilon\, B = (X + \epsilon c) + (\epsilon B - \epsilon c)$ belong to it, for each $\epsilon > 0$, and thus the thesis of the lemma is checked. $\square$

Lemma 2 *Let $\mathcal{D} \subset \mathcal{L}$ be a coherent set of desirable gambles satisfying the condition:*

$$X \in \mathcal{D}^* \text{ and } -X \notin \mathcal{D}^* \Rightarrow X \in \mathcal{D}. \tag{3}$$

Then, for any B subset of Ω such that $\overline{P}(B) > 0$., the following condition is also verified:

$$X \in \mathcal{D}^* \Rightarrow X + \epsilon\, B \in \mathcal{D},\ \forall\, \epsilon > 0.$$

Proof: Let us assume that $X + \epsilon \in \mathcal{D}$, $\forall\, \epsilon > 0$. Then, by the coherence of $\mathcal{D}$, $(X + \epsilon B) + \epsilon' = (X + \epsilon') + \epsilon\, B \in \mathcal{D}$, $\forall\, \epsilon, \epsilon' > 0$. So $X + \epsilon B \in \mathcal{D}^*$. To prove that this gamble is also in $\mathcal{D}$, we only have to prove that $-X - \epsilon B \notin \mathcal{D}^*$, i.e., there exists some $\epsilon'' > 0$ such that $-(X + \epsilon B) + \epsilon'' \notin \mathcal{D}$. Let us check it by contradiction. Let us suppose that $\epsilon'' - (X + \epsilon B) \in \mathcal{D}$, $\forall\, \epsilon'' > 0$. Then the gamble $\epsilon''' + \epsilon'' - \epsilon B = (X + \epsilon''') + (\epsilon'' - (X + \epsilon B))$ belongs to $\mathcal{D}$, for all $\epsilon'', \epsilon''' > 0$ by the coherence of $\mathcal{D}$. But the last assertion contradicts the assumption $\overline{P}(B) > 0$. $\square$

Theorem 3 *Let $\mathcal{D} \subset \mathcal{L}$ be a coherent set of desirable gambles and let B be a subset of the universe Ω. Let us assume that the following condition holds:*

$$(X \in \mathcal{D}^* \Rightarrow X + \epsilon\, B \in \mathcal{D},\ \forall\, \epsilon > 0). \tag{4}$$

Then,

$$\mathcal{P}_{\mathcal{D}_B} = (\mathcal{P}_{\mathcal{D}})_{|B},$$

where $(\mathcal{P}_{\mathcal{D}})_{|B}$ denotes the set of linear previsions

$$(\mathcal{P}_{\mathcal{D}})_{|B} = \{P(\cdot|B) \,:\, P \in \mathcal{P}_{\mathcal{D}} \text{ and } P(B) > 0\},$$

and, for each P with $P(B) > 0$, $P(\cdot|B)$ is defined as follows:

$$P(X|B) = \frac{P(BX)}{P(B)},\ \forall\, X \in \mathcal{L}.$$

Proof:

First, let us prove that $(\mathcal{P}_{\mathcal{D}})_{|B} \subseteq \mathcal{P}_{\mathcal{D}_B}$. If $Q \in (\mathcal{P}_{\mathcal{D}})_{|B}$, then $Q = P(.|B)$, where $P \in \mathcal{P}_{\mathcal{D}}$ and $P(B) > 0$.

If $X \in \mathcal{D}_B$, then either $X > 0$, and then it is verified $Q(X) \geq 0$, or $XB \in \mathcal{D}$. In the last case, as $P \in \mathcal{P}_{\mathcal{D}}$, we have that $P(XB) \geq 0$, and as $Q = P(.|B)$, then $Q(X) = Q(XB) = \frac{P(XB)}{P(B)} \geq 0$. Being $Q(X) \geq 0$ for any $X \in \mathcal{D}_B$, we can conclude that $Q \in \mathcal{P}_{\mathcal{D}_B}$.

To prove the other inclusion $\mathcal{P}_{\mathcal{D}_B} \subseteq (\mathcal{P}_{\mathcal{D}})_{|B}$, first consider that both are credal sets with probabilities which are 0 outside of B, then the inclusion can be obtained if we show that any linear restriction $P(X) \geq 0$ verified by probabilities in $(\mathcal{P}_{\mathcal{D}})_{|B}$ with $X(\omega) = 0, \forall \omega \in \Omega - B$, it is also verified by probabilities in $\mathcal{P}_{\mathcal{D}_B}$.

Assume that $X(\omega) = 0, \forall \omega \in \Omega - B$ and that $P(X) \geq 0, \quad \forall P \in (\mathcal{P}_{\mathcal{D}})_{|B}$. Then, we have that $Q(X|B) \geq 0, \quad \forall Q \in \mathcal{P}_{\mathcal{D}}$, with $Q(B) > 0$. As, $X(\omega) = 0, \forall \omega \in \Omega - B$, then $Q(X|B) = Q(XB)/Q(B) \geq 0, \quad \forall Q \in \mathcal{P}_{\mathcal{D}}, Q(B) > 0$. As, the inequality is trivially verified if $Q(B) = 0$, then we have that $Q(XB) \geq 0, \quad \forall Q \in \mathcal{P}_{\mathcal{D}}$.

If we add an amount ϵ to the gamble we obtain a desirable gamble: $XB + \epsilon \in \mathcal{D}, \quad \forall \epsilon > 0$, and by condition (4) we have that $XB + \epsilon B \in \mathcal{D}, \quad \forall \epsilon > 0$. By the definition of $\mathcal{D}_B$, we obtain that $XB + \epsilon B \in \mathcal{D}_B, \quad \forall \epsilon > 0$. This implies that $P(XB + \epsilon B) \geq 0, \quad \forall \epsilon > 0, \forall P \in \mathcal{P}_{\mathcal{D}_B}$ and therefore $P(XB) \geq 0, \forall P \in \mathcal{P}_{\mathcal{D}_B}$. As

$XB = X$, then the inequality $P(X) \geq 0$ is also verified by probabilities $P \in \mathcal{P}_{\mathcal{D}_B}$. $\square$

According to Lemmas 1 and 2 and Theorem 3, we derive the following corollary:

Corollary 4 *Let $\mathcal{D} \subset \mathcal{L}$ be a coherent set of desirable gambles and let B be an arbitrary subset of the universe Ω. Let us assume that one of the following conditions holds:*

1. $\underline{P}(B) > 0$.
2. $\overline{P}(B) > 0$ *and $\mathcal{D} \subset \mathcal{L}$ satisfies the restriction considered in Equation (3).*

Then:

$$\mathcal{P}_{\mathcal{D}_B} = (\mathcal{P}_{\mathcal{D}})_{|B}.$$

Remark 3.1 *When $\underline{P}(B) > 0$ the set of linear previsions $(\mathcal{P}_{\mathcal{D}})_{|B}$ can be written as follows:*

$$(\mathcal{P}_{\mathcal{D}})_{|B} = \{P(\cdot|B) \,:\, P \in \mathcal{P}_{\mathcal{D}}\},$$

since then condition $P(B) > 0$ is redundant.

This corollary represents the main result in this paper. First it shows the known fact that when $\underline{P}(B) > 0$, conditioning (in terms of credal sets) can be done by conditioning all the probability measures. The second thing is relative to conditioning when $\underline{P}(B) = 0$, but $\overline{P}(B) > 0$, in this case conditioning is not determined when we look at the associated credal set, but if we assume condition (3), then conditioning can be obtained in the associated credal set by conditioning all the probabilities with $P(B) > 0$, this conditioning was called *regular extension*. Condition (3) can be seen as a weaker version of Axiom D5, as here an almost desirable gamble X is also desirable when $-X$ is not almost desirable. If X and $-X$ are both almost desirable and we were accepting both of them as desirable, then we would obtain that the null gamble is desirable, and then the associated set would not be coherent. But, if X is almost desirable, but $-X$ is not, then it could be considered that we have some reasons to assume that X is desirable. Here, we have shown that this implies regular conditioning.

Example 2 *Assume that we have the credal set of Figure 2 and that we want to compute its conditional credal set to $B = \{\omega_1, \omega_2\}$ and that the points P with $P(B) = 1$ are the triangle base. In this case conditional gambles are those gambles X such that $X(\omega_3) = 0$ and the associated linear restrictions pass through the vertex opposite to the triangle base. When the credal set does not contain this vertex ($\underline{P}(B) > 0$), then there are desirable conditional gambles that determine that the conditional credal set is the thick segment represented in the basis and that is equal to the projection of all the probabilities in the credal set from the upper vertex (the projection of a probability P is its conditional probability $P(.|B)$). In other words, the set linear restrictions associated to the conditional gambles (passing through the upper vertex) that are strictly desirable (all the probabilities verify them and are not touching the credal set) as the one in the figure are enough to restrict the set on conditional probabilities to the segment in the figure.*

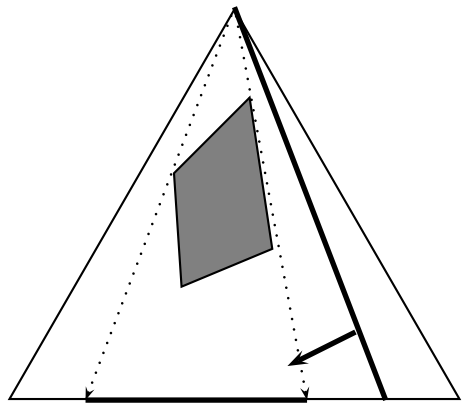

Figure 2: Conditional Gambles

However, when $\underline{P}(B) = 0$, then the upper vertex is in the credal set, as in Figure 3, and all the conditional gambles as the one depicted in the figure are touching the border of the credal set, and therefore their desirability is not determined by the credal set. The set of conditional desirable gambles could contain only the trivial gambles and then the conditional credal set is the full base (the natural extension *of the* generalized Bayes rule *[6]) or it could be a more restrictive one and include all the gambles with linear restrictions verified by the probabilities in the segment $\overline{AC}$ (the smallest possible conditional credal set: the regular extension).*

4 Introduction to Representation and Algorithms

A very important issue to make desirable gambles useful in practice is to determine an effective method to represent information and to develop algorithms able of working with this representation. In particular we would like to have procedures that have as input a set of gambles $\mathcal{F}$ and are able of carrying out the following basic reasoning tasks:

1. to determine whether the natural extension $\overline{\mathcal{F}}$ is coherent (i.e. $0 \notin \overline{\mathcal{F}}$),

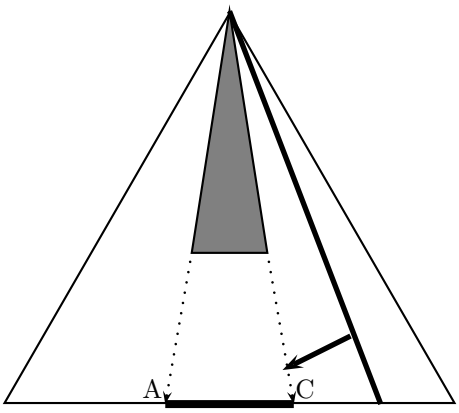

Figure 3: Conditional Gambles, $\underline{P}(B) = 0$

2. given X, to determine whether $X \in \overline{\mathcal{F}}$,
3. given X and $B \subset \Omega$, to compute $\underline{P}(X|B)$ and $\overline{P}(X|B)$ under $\overline{\mathcal{F}}$ when this set is coherent.

The second question is immediate to answer if we can solve the first one, as the following theorem shows.

Theorem 5 *If $\mathcal{F}$ is an arbitrary set of gambles such that $\overline{\mathcal{F}}$ is coherent, then $X \in \overline{\mathcal{F}}$ if and only if $\overline{\mathcal{F} \cup \{-X\}}$ is not coherent.*

Proof: If $X \in \overline{\mathcal{F}}$, then $X, -X \in \overline{\mathcal{F} \cup \{-X\}}$, and $X - X = 0 \in \overline{\mathcal{F} \cup \{-X\}}$. So this set is not coherent.

On the other hand, if $\overline{\mathcal{F} \cup \{-X\}}$ is not coherent, then $0 \in \overline{\mathcal{F} \cup \{-X\}}$. This set is equal to all the gambles $Y = \alpha.Z - \beta.X$, where $Z \in \overline{\mathcal{F}}$ and $\alpha, \beta \geq 0, \alpha > 0$ or $\beta > 0$. In particular, there must be α, β such that $0 = \alpha.Z - \beta.X$, where $Z \in \overline{\mathcal{F}}$. As $\overline{\mathcal{F}}$ is coherent, $\beta \neq 0$, and we obtain $X = \frac{\alpha}{\beta} Z$, and by Axiom D3, $X \in \overline{\mathcal{F}}$. $\square$

A coherent set of gambles $\mathcal{D}$ contains infinite gambles. If we want to represent them in a computer in order to manipulate them by means of algorithms, we need to determine a procedure to represent a coherent set of gambles $\mathcal{D}$ by means of a set $\mathcal{F}$ such that $\mathcal{D} = \overline{\mathcal{F}}$, and for representing the set $\mathcal{F}$ in some formal language. A basic issue is: to determine the type of sets $\mathcal{F}$ we are going to consider and the representation we are going to use. For sets of almost desirable gambles, we can start with a finite set of gambles $\mathcal{F}$ (which can be represented by enumerating the gambles in the set $\mathcal{F}$). This could also be done with sets of desirable gambles, but the capabilities of representation would be too limited, as the following example shows.

Example 3 *Assume that we know that $\overline{P}(B) = 0$, then the only possible set of desirable gambles representing this fact, should include all the gambles $\epsilon - B$ for any $\epsilon > 0$, but not the gamble in the limit $-B$. If $B \neq \Omega$, then $-B$ can be almost desirable without giving rise to an incoherent set. So this fact can be represented with a finite set of almost desirable gambles, but not with a finite set of desirable gambles.*

If we start with a finite set of gambles and compute its natural extension then some of the basic pieces of information can not be represented. In this paper, we want to point out a representation scheme which is not general enough for all the sets of desirable gambles, but which is enough for some of the most usual types of information and for which there are efficient algorithms in the literature.

Definition 1 *A* basic set of gambles *is a set of gambles $\mathcal{F}_{X,B} = \{X + \epsilon B : \epsilon > 0\}$, where X is an arbitrary gamble and $B \subseteq \Omega$. This set of gambles will be denoted as (X, B).*

When $B = \emptyset$, we have a single gamble, X. Otherwise, (X, B) is an infinite set with X in the limit.

The representation we propose is based in considering sets $\mathcal{F}$ given by the union of a finite family of basic sets of gambles: $(X_1, B_1), \ldots, (X_k, B_k)$.

With this system, $\underline{P}(X|B) = c$ is represented by means of $((X - c)B, B)$, i.e. in frame B, we are ready to pay $c - \epsilon$ to get reward $X(\omega)$, for any $\epsilon > 0$. $\overline{P}(X|B) = c$ is represented by means of $((c - X)B, B)$.

Coherence of the set of gambles generated by a finite set of basic gambles, $(X_1, B_1), \ldots, (X_k, B_k)$[1], is equivalent to the fact that the 0 gamble is not in the set of gambles generated by these gambles, which can be checked by showing that the following system in λ_i and ϵ has no solution:

$$\begin{array}{l} \sum_{i=1}^{k} \lambda_i (X_i + \epsilon B_i) \leq 0 \\ \lambda_i \geq 0, \quad \epsilon > 0 \end{array}$$

This is due to the fact that the set of gambles $\sum_{i=1}^{k} \lambda_i (X_i + \epsilon B_i)$ where $\lambda_i \geq 0, \quad \epsilon > 0$ is the set of gambles generated by the finite set of basic gambles by applying Axioms D3 and D4. So, we are checking whether the null gamble is contained in the natural extension.

An algorithm to solve this system is given by Walley, Pelessoni, and Vicig [8]. They start with a set of lower previsions of events, but they finally arrive to a system of this form, and propose an efficient algorithm to solve it, based on the resolution of a sequence of linear programming problems.

[1] We are considering coherence of the generated set of gambles and not the usual notion of coherence for conditional previsions which implies that none of the initial statements is strictly redundant.

To compute $\underline{P}(X|B)$ it is necessary to solve the following optimization problem (we are computing the supremum value of α such that $(X-\alpha)B$ is desirable in the natural extension of the basic gambles:

$$\begin{array}{l} \sup \alpha \\ s.t. \\ \sum_{i=1}^{k} \lambda_i (X_i + \epsilon B_i) \leq (X-\alpha)B \\ \epsilon > 0, \lambda_i \geq 0 \end{array}$$

This paper [8] also proposes algorithms to solve an optimization problem completely analogous to this one than can be easily adapted.

A basic question is whether there are simple sets of gambles which can not be covered with this representation. The following example shows a simple case in which there is no obvious solution by using this representation.

Example 4 *Consider $\Omega = \{\omega_1, \omega_2\}$ and the two gambles X, Y given by $X(\omega_1) = 1$, $X(\omega_2) = -1$ and $Y(\omega_1) = -1$, $Y(\omega_2) = 1$. Consider the set of gambles $\mathcal{F}$ given by $\epsilon_1 X + \epsilon_2 Y$, where $\epsilon_1, \epsilon_2 > 0$. This set of gambles is not coherent, as $X + Y = 0$ belongs to it. However, if we start with any representation $(X, B_1), (Y, B_2)$, then either $B_1 = B_2 = \emptyset$ with which we are adding X and Y to the set $\mathcal{F}$ (they were not initially in $\mathcal{F}$ as they can not be expressed as $\epsilon_1 X + \epsilon_2 Y$ with $\epsilon_1, \epsilon_2 > 0$) or if one of them, B_1 or B_2, is not empty, then the set generated by $(X, B_1), (Y, B_2)$ is coherent.*

The solution could be to start with more complex representations as $(X_1, \ldots, X_k)$ representing all the gambles $Z = \sum_{i=1}^{k} \epsilon_i X_i$ where $\epsilon_i > 0$, that we will call an *open set of gambles*. And then to work with sets of gambles which are generated by a finite family of open sets of gambles. However, the development of algorithms for coherence and inference is something to be done in the future, though it does not seem to be simple task.

5 Maximal Sets of Gambles

In this section we will investigate maximal coherent sets of gambles. These sets of gambles represent a *complete* uncertain knowledge: adding a single more gamble will give rise to an incoherent set. The associated credal sets will be linear previsions (probability measures). But, we will also be able of associating finite sequences of probability measures similar to the ones considered by Coletti and Scozzafava [2].

Definition 2 *We will say that a set of gambles $\mathcal{D}$ is* maximal *if it is coherent and there does not exist any $X \notin \mathcal{D}$ such that $\overline{\mathcal{D} \cup \{X\}}$ is coherent.*

Lemma 6 *If $\mathcal{D}$ is coherent and $-X \notin \mathcal{D}$, $X \neq 0$, then $\overline{\mathcal{D} \cup \{X\}}$ is coherent.*

Proof: Let us check it by contradiction. Let us suppose that $\overline{D \cup \{X\}}$ is not coherent. Then there exists a collection of non-negative numbers $c_1, \ldots, c_n, c_{n+1}$ such that $\sum_{i=1}^{n} c_i X_i + c_{n+1} X = 0$, where some of the c_i's is non zero. Furthermore, according to the coherence of $\mathcal{D}$, $c_{n+1} \neq 0$. And, as $X \neq 0$, some of the $c_i, i = 1, \ldots, n$ is also different from 0. Thus, $-X$ can be written as follows: $-X = \sum_{i=1}^{n} \frac{c_i}{c_{n+1}} X_i$. Then, by the coherence of $\mathcal{D}$, $-X$ belongs to it, and we get a contradiction. $\square$

Theorem 7 *A coherent set of gambles $\mathcal{D}$ is maximal if and only if $X \in \mathcal{D}$ xor $-X \in \mathcal{D}$, for all $X \in \mathcal{L}$, $X \neq 0$.*

Proof: Let us suppose that $\mathcal{D}$ is maximal and $X \notin \mathcal{D}$. Then, by definition, $\overline{(\mathcal{D} \cup \{X\})}$ is not coherent. Thus, according to Lemma 6, $-X$ must belong to $\mathcal{D}$. On the other hand, if for any $X \in \mathcal{L}$, $X \in \mathcal{D}$ xor $-X \in \mathcal{D}$, then $\mathcal{D}$ is maximal, as if $X \notin \mathcal{D}$, then $-X \in \mathcal{D}$ and $\overline{\mathcal{D} \cup \{X\}}$ can not be coherent. $\square$

Lemma 8 *If $\mathcal{D}$ is maximal then $\mathcal{D}_B$ is maximal for all $B \subseteq \Omega$, $B \neq \emptyset$.*

Proof: It is trivially derived from Theorem 7. $\square$

Lemma 9 *Let $\mathcal{D}$ be a maximal set of gambles and let $\underline{P}$ and $\overline{P}$ be respectively the lower and the upper previsions associated to it. Then $\underline{P}(B) = \overline{P}(B)$, $\forall B \subseteq \Omega$.*

Proof: Let us prove it by contradiction. Let us suppose that there exists some $B \subseteq \Omega$ such that $\underline{P}(B) < \overline{P}(B)$. Then, for all $p \in (\underline{P}(B), \overline{P}(B))$, $B - p \notin \mathcal{D}$ and $-(B-p) = p - B \notin \mathcal{D}$. According to Theorem 7, $\mathcal{D}$ cannot be maximal. $\square$

If we have a sequence of nested sets $\Omega = C_0 \supset C_1 \supset \cdots \supset C_n = \emptyset$, and $B \subseteq \Omega$, then the *layer* of B with respect to this sequence, will be the minimum value of i such that $B \cap (C_i \setminus C_{i+1}) \neq \emptyset$. It will be denoted by layer(B).

Theorem 10 *If $\mathcal{D}$ is maximal then there is a sequence of nested sets $\Omega = C_0 \supset C_1 \supset \cdots \supset C_n = \emptyset$ and a sequence of probability measures $P_0, \ldots, P_{n-1}$ satisfying the following conditions:*

1. *for each probability P_i, $P_i(C_i \setminus C_{i+1}) = 1$, $P_i(\omega) > 0$ for any $\omega \in C_i \setminus C_{i+1}$,*
2. *for each $A \subseteq B \subseteq \Omega$, if $i = \text{layer}(B)$, then $\underline{P}(A|B) = \overline{P}(A|B) = P_i(A|B)$, where $\underline{P}(A|B)$ and $\overline{P}(A|B)$ are the lower and upper probabilities computed from $\mathcal{D}_B$.*

Proof: According to Lemma 9, the lower and the upper probabilities associated to $\mathcal{D}$ do coincide. In other words, the class $\mathcal{P}_{\mathcal{D}}$ is a singleton (it is determined by an additive probability measure P on Ω). Let $C_1 \subsetneq \Omega$ be the subset of elements of probability 0 $C_1 = \{\omega \in \Omega : P(\{\omega\}) = 0\}$. If $C_1 \neq \emptyset$, according to Lemma 8, $\mathcal{D}_{C_1}$ is maximal. Based again on Lemma 9, it induces a probability measure on C_1, P_1. We can repeat the same process again and get a strictly decreasing finite sequence of nonempty sets C_i and an associated finite family of probability measures P_i. (Note that, after a finite sequence of n steps, the set C_n will be the empty set and the process is finished.)

On the other hand, if $A \subseteq B \subseteq \Omega$ and i is the layer of B, then we have that $B \subseteq C_i$ and $P_i(B) > 0$ (remember that C_{i+1} is the subset of C_i given by the $\omega \in C_i$ such that $P_i(\omega) = 0$). Probability P_i is defined in C_i and as it is associated to $\mathcal{D}_{C_i}$ and this set is maximal, we have that $\underline{P}(E|C_i) = \overline{P}(E|C_i) = P_i(E)$ for any $E \subseteq C_i$. As $P_i(B) > 0$, then its lower conditional probability is greater than 0, and by Corollary 4, the conditional probability can be computed by conditioning in the associated credal set, obtaining the desired result:

$$P_i(A|B) = \underline{P}(E|C_i \cap B) = \overline{P}(A|C_i \cap B) = \underline{P}(A|B) = \overline{P}(A|B)$$

□

This theorem shows the great similarity between maximal coherent gambles and the sequence of probabilities associated to a coherent set of conditional assessments given by Coletti and Scozzafava [2]. The layer of B is also the minimum value of i for which $P_i(B) > 0$, and therefore is the equivalent concept to the *zero layer* of B proposed by these authors. However, there are some differences between the two models as we will show later: we can have the same sequence of probabilities associated to different maximal coherent sets of desirable gambles.

In the following we show that any coherent set of gambles is the intersection of a family of coherent maximal sets of gambles. First, we need a technical result.

Lemma 11 *If $\mathcal{D}$ is coherent and $-X, X \notin \mathcal{D}$ and $X \neq 0$, then $\mathcal{D}^{+X} = \overline{(\mathcal{D} \cup \{X\} \cup \{-X + Y : Y \in \mathcal{D})}$ is coherent.*

Proof: If this set is not coherent, then we have that there are $\alpha_1, \alpha_2, \alpha_3 \geq 0$, and $Y_1, Y_2 \in \mathcal{D}$ such that $\alpha_1 Y_1 + \alpha_2 X + \alpha_3(-X + Y_2) \leq 0$, and at least one of the α_i is not equal to 0.

From this inequality we have that: $\alpha_1 Y_1 + \alpha_3 Y_2 \leq (\alpha_3 - \alpha_2)X$.

First, notice that α_1, α_3 can not be both equal to 0, because otherwise, $0 \leq (-\alpha_2)X$, and as $X \neq 0$ and $\alpha_2 \neq 0$, we have that $-X \in \mathcal{D}$, which is in contradiction with the fact that $-X, X \notin \mathcal{D}$.

Then, at least one of the values α_1, α_3 is different from 0, and thus $\alpha_1 Y_1 + \alpha_3 Y_2 \in \mathcal{D}$.

Three situations are now possible:

- $\alpha_3 = \alpha_2$, which is in contradiction with the fact that $\mathcal{D}$ is coherent.
- $(\alpha_3 - \alpha_2) > 0$, which is in contradiction with the fact that $\mathcal{D}$ is coherent and $X \notin \mathcal{D}$.
- $(\alpha_2 - \alpha_3) > 0$, which is in contradiction with the fact that $\mathcal{D}$ is coherent and $-X \notin \mathcal{D}$.

In any case, we arrive to a contradiction, so $\mathcal{D}^{+X}$ must be coherent. □

Theorem 12 *Let $\mathcal{D}$ be a coherent set of gambles. Then, there exists at least one maximal set of gambles containing it.*

Proof:

Let us start with a coherent set and then, repeat the following process:

1. If for any gamble X ($X \neq 0$), we have that $X \in \mathcal{D}$ or $-X \in \mathcal{D}$, then $\mathcal{D}$ is maximal and the procedure stops.
2. Select a gamble X such that $-X, X \notin \mathcal{D}$ and $X \neq 0$.
3. Transform $\mathcal{D}$ by making it equal to $\mathcal{D}^{+X}$. By Lemma 11, this new set is coherent and contains to the old $\mathcal{D}$.
4. Go to step 1.

The main point of this procedure is that it arrives to a maximal coherent set after a finite number of steps. This result is based on the fact that if in the first $k+1$ loops of this process we select gambles $X_1, X_2, \ldots, X_{k+1}$, then these gambles are linearly independent. This fact is obtained by proving that after having added $X_1, \ldots, X_k$, then for any linear combination of these gambles $Y = \sum_{i=1}^{k} \alpha_i X_i$, and $Y \neq 0$, we have that either $Y \in \mathcal{D}$ or $-Y \in \mathcal{D}$. So, in step 2, we have to select a gamble which is linearly independent of the previously selected ones.

This is going to be proved by induction in k. For $k = 1$, $Y = \alpha_1 X_1$. Then if $\alpha_1 > 0$, $Y \in \mathcal{D}$, and if $\alpha_1 < 0$, then $-Y \in \mathcal{D}$. α_1 can not be equal to 0 because $Y \neq 0$.

Now, assume that it is true for the first k gambles $X_1, \ldots, X_k$, and let us prove it for $X_1, X_2, \ldots, X_{k+1}$.

Assume, $Y = \sum_{i=1}^{k+1} \alpha_i X_i$. Let us denote by $Z = \sum_{i=1}^{k} \alpha_i X_i$.

If $\alpha_i = 0$ for all $1 \leq i \leq k$, then $Y = \alpha_{k+1} X_{k+1}$ and we are in a situation similar to the case $k = 1$.

If some α_i with $i \leq k$ is different from 0, then by induction, we have that either Z or $-Z$ is in $\mathcal{D}$, after adding $X_1, \ldots, X_k$.

We have that $Y = Z + \alpha_{k+1} X_{k+1}$. The following situations are possible:

- $\alpha_{k+1} = 0$, then $Y = Z$ and we have that either Y or $-Y$ is in $\mathcal{D}$.
- $\alpha_{k+1} > 0$ and $Z \in \mathcal{D}$, then by coherence $Y \in \mathcal{D}$.
- $\alpha_{k+1} > 0$ and $-Z \in \mathcal{D}$, then $-Y = -Z - \alpha_{k+1} X_{k+1}$, and by the way we compute $\mathcal{D}^{+X_{k+1}}$ in which we add any gamble $-X_{k+1} + U$, and therefore any gamble $-\alpha_{k+1} X_{k+1} + U$ where $U \in \mathcal{D}$, we have that $-Y \in \mathcal{D}$.
- $\alpha_{k+1} < 0$ and $-Z \in \mathcal{D}$, then by coherence $-Y \in \mathcal{D}$.
- $\alpha_{k+1} < 0$ and $Z \in \mathcal{D}$, then $Y = \alpha_{k+1} X_{k+1} + Z \in \mathcal{D}$ after replacing $\mathcal{D}$ by $\mathcal{D}^{+X_{k+1}}$.

As, we always choose a gamble that is linearly independent of the previous one, and Ω being finite, the dimension of $\mathcal{L}$ as a linear space is finite, and so the process has to stop after a finite number of steps. $\square$

Theorem 13 *Let $\mathcal{D}$ be a coherent set of gambles. Then $\mathcal{D} = \cap_{i \in I} \mathcal{D}_i$, where $\{\mathcal{D}_i : i \in I\}$ is the class of maximal sets of gambles containing $\mathcal{D}$.*

Proof: We only have to check the inclusion $\cap_{i \in I} \mathcal{D}_i \subseteq \mathcal{D}$. We will prove it by contradiction. Let us suppose that $X \in \cap_{i \in I} \mathcal{D}_i \setminus \mathcal{D}$. Then, by Lemma 6, $\overline{\mathcal{D} \cup \{-X\}}$ is coherent. By Theorem 12 there exists at least one maximal set of gambles containing $\overline{\mathcal{D} \cup \{-X\}}$. This maximal set coincides with one of the $\mathcal{D}_i$, for some $i \in I$. Then there exists some $i \in I$ such that $-X \in \mathcal{D}_i$. It contradicts the assumption of coherence of $\mathcal{D}_i$. $\square$

This theorem can be the basis to obtain a representation of gambles analogous to the credal sets for sets of almost desirable gambles. Now, a coherent set of gambles can be expressed as a family of maximally coherent sets of gambles, each one of them has an associated sequence of probability measures. There are important problems to be solved. One of them is that a maximally consistent coherent set of gambles is not exactly equivalent to a sequence of probability measures as the following example shows.

Example 5 *Assume that $\Omega = \{\omega_1, \omega_2\}$ and the probability given by $P_0(\omega_1) = P_0(\omega_2) = 0.5$ (only one probability in the sequence). It is clear that any gamble with $X(\omega_1) + X(\omega_2) > 0$ should be desirable. But, this probability does not determine whether the gamble $Y(\omega_1) = 1, Y(\omega_2) = -1$ is desirable. We can have a coherent set in which neither Y nor $-Y$ is desirable, another coherent set in which Y is desirable, and other one in which $-Y$ is desirable. Only the two last ones are maximal.*

An alternative model that allows to establish a correspondence between maximally coherent sets of gambles and sequences of probability measures is obtained by making the consistency Axiom D1 stronger, modifying it to the following version:

D1". If $X \in \mathcal{D}$, then there is $\epsilon > 0$, such that $-X + \epsilon\,\mathrm{supp}(X) \notin \mathcal{D}$.

where $\mathrm{supp}(X)$, the support of X, is the set of $\omega \in \Omega$ such that $X(\omega) \neq 0$.

This consistency condition is stronger than Axiom D1, as this axiom was assuming that we can not have X and $-X$ as desirable. D1" implies that the null gamble is not desirable. This axiom says that we can not have X as desirable if $-X$ is the limit of desirable gambles with the same support. This is a kind of a minimum of separation between X and $-Y$, if both X and Y are desirable and have the same support. The support is necessary in the condition, as if we had only considered $-X + \epsilon$ (as in strict desirability axioms [6, Section 3.7.8]), then it would have become a too strong condition. Imagine that $P_0(B) = 0$, then as $P_0(B) = 0$ we have that $-B + \epsilon$ is also desirable for any $\epsilon > 0$. But we have that $B \in \mathcal{D}$. So, the separation condition without considering the support would not be fulfilled.

The following theorem shows that a sequence of probability measures as the one generated in Theorem 10 can always be represented by means of a maximally coherent set of gambles among those satisfying D1" and that this set is unique.

Theorem 14 *If we have a sequence of nested sets $\Omega = C_0 \supset C_1 \supset \cdots \supset C_n = \emptyset$ and a sequence of probability measures $P_0, \ldots, P_{n-1}$ satisfying condition:*

- *for any i, $P_i(C_i \setminus C_{i+1}) = 1$, and $P_i(\omega) > 0, \quad \forall \omega \in C_i \setminus C_{i+1}$,*

then the set of gambles $\mathcal{D} = \{X : P_i(X) > 0$, where $i = \mathrm{layer}(\mathrm{supp}(X))\}$ is the only maximally

coherent set of desirable gambles among those satisfying Axiom D1" and that for any i the credal set associated to $\mathcal{D}_{C_i}$ contains only probability measure P_i.

Proof: First, it is easy to prove that this set of gambles satisfies all the axioms for coherence including Axiom D1". Considering $P_i(X) > 0$, we have that $P_i(\text{supp}(X)) > 0$, $P_i(-X + \epsilon\,\text{supp}(X)) = -P_i(X) + \epsilon < 0$ if we choose $\epsilon > 0$ small enough. Therefore, there is an $\epsilon > 0$ such that $-X + \epsilon\,\text{supp}(X) \notin \mathcal{D}$.

It is immediate to prove that P_i is the credal set associated to $\mathcal{D}_{C_i}$, as there can not be a probability Q different to P_i defined on C_i for which $Q(X) > 0$ for all $X \in \mathcal{D}_{C_i}$.

On the other hand, this set is unique: if $\mathcal{D}'$ is such that for any i the credal set associated to $\mathcal{D}'_{C_i}$ is the probability P_i, then, if X is a gamble and $i = \text{layer}(\text{supp}(X))$, then $\text{supp}(X) \subseteq C_i$ and:

- if X is such that $P_i(X) > 0$, then $X \in \mathcal{D}'$,
- if X is such that $P_i(X) < 0$, then $X \notin \mathcal{D}'$,
- if X is such that $P_i(X) = 0$, then as there is $\omega \in C_i \setminus C_{i+1}$ such that $X(\omega) > 0$ and $P_i(\omega) > 0$, we have that for any $\epsilon > 0$, $P_i(-X+\epsilon\,\text{supp}(X)) = P_i(-X) + \epsilon = \epsilon > 0$ then we have that $-X + \epsilon\,\text{supp}(X) \in \mathcal{D}'_{C_i} \subseteq \mathcal{D}'$. By Axiom D1", $X \notin \mathcal{D}'$.

As a consequence, $\mathcal{D}'$ obeys the same criteria than $\mathcal{D}$ to determine whether a gamble belongs to it ($P_i(X) > 0$) and thus $\mathcal{D} = \mathcal{D}'$.

The fact that $\mathcal{D}$ is maximal is a consequence of the uniqueness. $\square$

6 Conclusions

In this paper we have presented some basic concepts under the light of desirability. We have tried to show that this approach can shed light on some important notions in imprecise probability, such as conditioning. It can be useful for showing the relationships with other approaches, such as as probabilistic coherence [2]. There is important work to do, mainly on the algorithmic side. Here we have shown some existing algorithms which can be directly applied to some restricted forms of coherent sets, but it is necessary to determine whether this restriction is too severe to leave out some important real situations. Also, it would be interesting to determine some extra axioms under which the representation based on what we have called basic sets of gambles is enough to cover any possible set of desirable gambles.

As we have mentioned, Moral [4] studied the concept of epistemic irrelevance and independence taking desirability as basis. An important problem for the future is how to use graphical models to represent and use epistemic independence assessments in the computation of conditional sets of desirable gambles.

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