Dutch Books and Combinatorial Games

Peter Harremoës
Centrum Wiskunde & Informatica (CWI), The Netherlands
P.Harremoes@cwi.nl

Abstract

The theory of combinatorial games (like board games) and the theory of social games (where one looks for Nash equilibria) are normally considered two separate theories. Here we shall see what comes out of combining the ideas. J. Conway observed that there is a one-to-one correspondence between the real numbers and a special type of combinatorial games. Therefore the payoffs of a social games are combinatorial games. Probability theory should be considered a safety net that prevents inconsistent decisions via the Dutch Book Argument. This result can be extended to situations where the payoff function yields a more general game than a real number. The main difference between number-valued payoff and game-valued payoff is that the existence of a probability distribution that gives non-negative mean payoff does not ensure that the game will not be lost.

Keywords. Combinatorial game, Dutch Book Theorem, exchangable sequences, game theory, surreal number.

1 Introduction

The word game in mathematics has two different meanings. The first type of games are the *social games* where a number of agents at the same time have to make a choice and where the payoff to each agent is a function of all agents' choices. Each agent has his own payoff function. The question is how the agents should choose in order to maximize their own payoff. In general the players may benefit by making coalitions against each other. This kind of game theory has found important applications in social sciences and economy. A special class of these social games are the two-person zero-sum games where collaboration between the agents makes no sense.

The second type of games are the *combinatorial games.* These are mathematical models of board games. These games are the ones that people find interesting and amusing. Games that people play for amusement often involve an element of chance, generated by, for instance, dice, but the combinatorial games are by definition the ones that do not contain this element. Therefore they are sometimes called *games of no chance* [15]. Examples from this category are chess, nim, nine-mens-morris, and go. Combinatorial game theory has been particularly successful in the analysis of impartial games like nim [5] and has lead to a better understanding of endgames in go [3, 4, 15].

The Dutch Book Theorem is important in our understanding of imprecise probabilities. The Dutch Book Theorem was first formulated and proved by F. P. Ramsay in 1926 (reprinted in [16]) and later independently by B. de Finetti [8], who used it as an argument for a subjective interpretation of probabilities. Since the original formulation of the Dutch Book Theorem most of the research has been in the direction of more subjective versions. As it is normally formulated, the theorem relies on the concept of a *real-valued payoff* function. One may think of an outcome of the payoff function as money but the uniform mean of having £ 1.000.000 and having £ 0 is having £ 500.000. Most people have a very clear preference for having £ 500.000 rather than an unknown amount of money with mean £ 500.000. Instead one may think of the payoff as a more subjective notion of *value,* but this is also a highly debatable concept and one may actually consider money as our best attempt to quantify value. Savage showed that the concept of value and payoff function can be replaced by the concept of preference, so that a coherent set of preferences corresponds to the existence of a payoff function and a probability measure. This line of research has been followed up by many other researchers [6, 17]. All those studies involve some subjective notion of value or preference.

In order to better understand the Dutch Book Theorem it is desirable to see how the theory would look in an environment where a subjective notion of value plays no role. In this study we replace the normal payoff functions by game-valued functions. There are several reasons why this is of interest:

- A real-valued payoff function is a special case of a game-valued payoff function.
- The theory of probability has its origin in the study of games involving chance.
- Social game theory and combinatorial game theory may mutually benefit from a closer interaction.
- One can often get insight into a special case by the study of its generalizations.

With a game-valued payoff function the players in a social game have to play a certain combinatorial game that depends on their decisions and/or on some random event. This setup may seem quite contrived, but many board games that involve chance are of this form.

Example 1 *In chess it is normally considered a slight advantage to play white. Therefore one normally randomly selects who should play white and who should play black.*

Example 2 *M. Ettinger has developed an interesting version of combinatorial game theory where after each move a coin is flipped to determine who is going to play next [9].*

Actually any board game involving chance may be considered as an example. It will be the subject of a future paper how to take advantage of a combined probabilistic and combinatorial game approach for some specific board games. In this short note we shall focus entirely on how we should formulate or reformulate the Dutch Book Theorem when the payoffs are combinatorial games.

Social games and combinatorial games are built on quite different ideas and many scientists only know one of the types of game theory. There have only been few attempts to combine the two types of game theory [9, 22]. In this exposition we will assume that the reader has basic knowledge about social games such as two-person zero-sum games. Nevertheless we have to repeat some of the elementary definitions from social game theory in order to fix notation and, in particular, to avoid confusion with similar but slightly different concepts from combinatorial game theory.

Our main result is that it is possible to formulate versions of the Dutch Book Theorem for game-valued payoff functions, but there will be some important modifications of the theorem. For instance our probability distributions will not always be real-valued. In our approach the focus is on order structure (induced by games) and its relation to decision theory. A somewhat orthogonal approach was taken in [13] where the probabilities were elements of a metric space with no order structure.

2 Combinatorial games

The theory of combinatorial games was developed by J. Conway as a tool to analyze board games [5, 7]. A short and more careful exposition can be found in [18]. In a board game the players *alternate* in making moves. Each move changes the configuration of the pieces on the board to some other configuration but only certain changes are allowed. It is convenient to call the two players *Left* and *Right*. We shall often consider different board configurations as different games. If G denotes a game, i.e. a certain configuration then the game is specified by the configurations G^L that Left is allow to move to and the configurations G^R that Right is allowed to move to, and we write $G = \left\{G^L \mid G^R\right\}$. Note that we have not told who is playing first, and therefore we have to describe it from both Left's and Right's perspective. Now the point is that G^L and G^R are sets of games, so a game is formally a specification of two sets of games. In a board game it is nice to have many options to choose among and bad if there are only few options. The worst case for Left is if there are *no options left* and in this case we say that Left has *lost* the game. So Left has lost the game if he is to move next and G^L is empty. Similar Right loses the game if it is Right to move and G^R is empty. The rules of many board games can be modelled in this way.

Example 3 (Games illustrated in Figure 1.) *The game* $\{\varnothing \mid \varnothing\}$ *is a boring one. The one to move first loses this game. This game is denoted* 0.

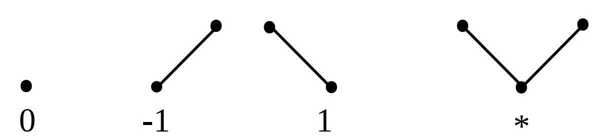

Figure 1: Games can be illustrated by game trees. Options for Left are illustrated by left slanting edges, and options for Right are illustrated by right slanting edges. Here are the simplest ones. In more complicted games there may be several left or right slanting edges from each node.

The game $\{\varnothing \mid 0\}$ *is lost by Left if Left has to move first. If Right goes first Right has to choose* 0*. Now it is Left to move but this is a losing position for the one who is going to move, so poor Left loses. Thus Right always wins the game* $\{\varnothing \mid 0\}$ *. This game is denoted* -1.

The game $\{0 \mid \varnothing\}$ *is lost by Right if Right has to move first. If Left goes first Left has to choose* 0*. Now it is Right to move but this is a losing position for the one who is going to move, so now Left is happy again because he wins. Thus Left always wins the game* $\{0 \mid \varnothing\}$ *. This game is denoted* 1.

Similarly we see that $\{0 \mid 0\}$ *is won by the player that moves first. This game is called* star *and is denoted* $*$*. In Japanese go terminology such a position is called* dame*.*

Here we shall use the following recursive definition of a game.

Definition 1 *A* game *is a pair* $\left\{G^L \mid G^R\right\}$ *where* G^L *and* G^R *are sets of already defined games.*

The *status* of a game G can be classified according to who wins if both players play optimally. We define

$$
\begin{array}{ll}
G = 0, & \text{if second player wins;}\\
G < 0, & \text{if Right wins whoever plays first;}\\
G > 0, & \text{if Left wins whoever plays first;}\\
G \parallel 0, & \text{if first player wins.}
\end{array}
$$

For a game G we can reverse the role of Left and Right and call this the *negative of the game*. Formally we use the following recursive definition.

$$- \left\{G^L \mid G^R\right\} = \left\{-G^R \mid -G^L\right\}.$$

Left and Right can play two games in parallel. In every round each player should make a move in one of the games of his own choice. Perhaps there are urgent moves to be made in both games so the players have to prioritize in which game it is most important to make the move. Several games played in parallel is called the *sum of the games*, and many positions in actual board games can be understood as sums of sub-games. Combinatorial game theory is essentially the theory of how to prioritize your moves in a board game that has the structure of a sum of independent sub-games. Formally the sum of the games G and H is defined recursively by

$$G + H = \left\{\left(G^L + H\right) \cup \left(G + H^L\right) \mid \left(G^R + H\right) \cup \left(G + H^R\right)\right\}.$$

The sum of games is normally illustrated by the disjoint union of the game trees of the individual games. The game $G - H$ is by definition the game $G + (-H)$.

Now, we are able to define what it should mean that two games are equal. We write $G = H$ if $G - H = 0$, i.e. second player wins $G - H$. One can define $G > H$, $G < H$, and $G \parallel H$ in the same way. We say that G and H are *confused* if $G \parallel H$. One can prove that $G = H$ if and only if $G + K$ and $H + K$ have the same status for any game K.

With these operations the class of games has the structure as a partially ordered Abelian group. Any Abelian group is a module over the ring of integers with multiplication defined as follows. If n is a natural number we define $n \cdot G$ by

$$\overbrace{G + G + \cdots + G}^{n \text{ times}}.$$

If $n = 0$ then $0 \cdot G$ is by definition equal to 0. If n is a negative integer we define $n \cdot G$ to be equal to $(-n) \cdot (-G)$.

The equation $2 \cdot G = 0$ has $G = 0$ as solution, but $G = *$ is also a solution. Therefore there is in general no unique way of defining multiplication of a game by $1/2$, and the same holds for other non-integers. From this point of view it is surprising that all dyadic fractions (rational numbers of the form $n/2^m$) can be identified with games. One way of doing it goes as follows.

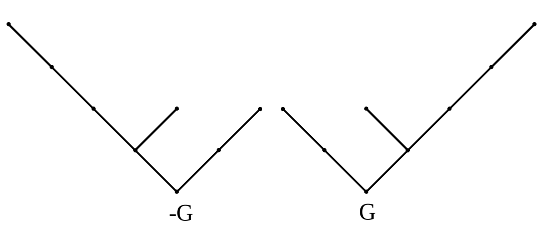

Figure 2: The game tree of $-G$ is simply the mirror image of the game tree of G.

3 Numbers may be identified with games

J. Conway discovered that all real numbers can be identified with games but his construction will lead to a larger class of numbers called the *surreal numbers* (or Conway numbers). The surreal numbers were first described in a mathematical novel by D. Knuth [14], and later in much detail by J. Conway [7]. For newer and more complete descriptions we refer to [1, 11].

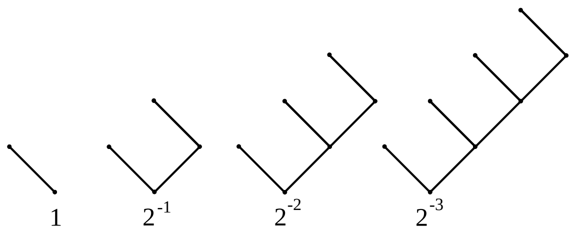

Figure 3: Some dyadic fractions.

We have already defined the game 1 so the integer n is identified with the game $n \cdot 1$. The game $\{0 \mid 1\}$ satisfies

$$2 \cdot \{0 \mid 1\} = 1.$$

Hence the 2^{-1} can be identified with the game $\{0 \mid 1\}$. In general the game $\{0 \mid 2^{-m}\}$ satisfies

$$2 \cdot \left\{0 \mid 2^{-m}\right\} = 2^{-m}$$

so the fraction $2^{-(m+1)}$ can be identified with the game $\{0 \mid 2^{-m}\}$ (see Figure 3). Thus the fraction $n/2^m$ can be identified with the game $n \cdot 2^{-m}$. In this way any dyadic fraction can be identified with a game.

A real number can be identified with a Dedekind section in the group of dyadic fractions. In other words, a real number r, can be identified with the partition of the dyadic fractions into the sets

$$A = \{n \cdot 1/2^m < r \mid m, n \in \mathbb{N}\},$$
$$B = \{n \cdot 1/2^m > r \mid m, n \in \mathbb{N}\}.$$

Now, A and B can be identified with sets of games and therefore $\{A \mid B\}$ is a game. When r is a real number that is not a dyadic fraction, it can be identified with the game $\{A \mid B\}$. At this step one has to check that the structure of the real numbers as an ordered group is preserved under the embedding but this turn out to be the case [7].

We have seen that real numbers may be identified with games, but combining the definition of a game with the idea of a Dedekind section leads to the much larger class of numbers called the *surreal numbers*. Formally a surreal number is a game of the form $\{A \mid B\}$ where A and B are sets of (already constructed) surreal numbers such that $a < b$ for $a \in A$ and $b \in B$. That means that a surreal number can always be played as a combinatorial game.

Example 4 *The first transfinite ordinal number ω is identified with the game $\{\mathbb{N} \mid \varnothing\}$. The equation $\omega - \omega = 0$ makes no sense in Cantor's arithmetic for transfinite ordinals or cardinals, but if we identify ω with a game the equation makes sense, because we have*

$$\omega - \omega = \{1, 2, 3, \cdots \mid \varnothing\} + \{\varnothing \mid -1, -2, -3, \cdots\}.$$

This game is essentially like "my father has more money than your father" and most children soon experience that one should not start in such a game. It is clear that ω should not be interpreted as an amount but is better understood as a huge set of options. Conway identified all Cantor's ordinal numbers with surreal numbers, but Cantor and Conway use different *additive structures so the identification is somewhat problematic. For instance Conway's addition is commutative but Cantor's addition of ordinal numbers is not. Here we shall use ω as a symbol for a game rather than an ordinal in Cantor's sense.*

Formally the surreal numbers are constructed by (transfinite) recursion. It starts with the number $0 = \{\varnothing \mid \varnothing\}$. In each recursion step one adds new surreal numbers to the ones already constructed. Addition and multiplication extend to surreal numbers and with these operations the surreal numbers are a maximal ordered field. Although the definition of surreal multiplication is relevant for the next two sections we cannot present the definition in this short note but have to refer to [7,18]. For most computations surreal numbers are not different from real numbers but the topology is different.

A game G is said to be *infinitesimal* if $-2^{-m} \leq G \leq 2^{-m}$ for all natural numbers m. The number $1/\omega$ is an example of an infinitesimal number that is positive. Between any two different real numbers there are more than continuously many surreal numbers, and the intersection of the intervals $[-2^{-m}; 2^{-m}]$ contains infinitely many *infinitesimal numbers*. Formally there are so many surreal numbers that they do not form a set but a class.

4 Surreal probabilities and payoffs

Here we will introduce a version of the *Dutch Book Theorem* for surreal payoff functions. Because of the somewhat different topology of the surreal numbers, we have to be a little careful in the formulation and proof of the Dutch Book Theorem. In particular some of the standard methods for proving these results like the Hahn-Banach theorem and the separation theorem for convex sets, do not hold in their normal formulation when we are using surreal numbers. Those used to to non-standard analysis may note that what we are doing is essentially to veify that our result may be formulated in first order language.

The setup is as follows. Alice wishes to make a bet on an outcome $a \in A$. A bookmaker $b \in B$ offers the surreal payoff $g(a, b)$ (positive or negative) if the outcome of a random event is $a \in A$. Thus $(a,b) \to g(a,b)$ can be considered as a matrix when A and B are finite sets. Alice should reject to play with a bookmaker b if Alice thinks that the payoff function $a \to g(a,b)$ is not favorable. For simplicity we shall assume that Alice accepts the payoff functions offered by the bookmakers $b \in B$. We recall that a surreal number is a game so if the outcome is a and the bookmaker is b then Alice has to play the game $g(a,b)$ against the bookmaker with Alice playing Left and the bookmaker playing Right.

By a *portfolio* we shall mean a probability vector $Q = (q_b)_{b\in B}$ on B. In this section will allow the portfolio to have surreal values. Such a portfolio is described by the payoff function

$$a \to \sum_{b\in B} q_b \cdot g(a,b), \tag{1}$$

A *Dutch book* is a portfolio such that (1) is negative for all $a \in A$, i.e. the portfolio game will be lost by Alice for any value of $a \in A$.

We assume that one of the bookmakers b_0 offers a payoff function $g(a,b_0) = 0$ for all $a \in A$ (b_0 acts like a bank with interest rate 0). Let Q be a portfolio and assume that there exists a Dutch book Q'. If Q has B as support then $q_{\min} = \min_{b\in B} q_b > 0$ and the payoff is

$$\sum_{b\in B} q_b \cdot g(\cdot,b) =$$

$$\sum_{b\in B} (q_b - q_{\min} \cdot q_b') \cdot g(\cdot,b) + q_{\min} q_b \sum_{b\in B} q_b' \cdot g(\cdot,b) <$$

$$\sum_{b\in B} (q_b - q_{\min} \cdot q_b') \cdot g(\cdot,b) + \left(q_{\min} \sum_{b\in B} q_b'\right) \cdot g(\cdot,b_0).$$

Hence Alice should reject to play with at least one of the bookmakers. If no Dutch book exists the set of payoff functions is said to be *coherent*. The notion of convexity will be used, and in this section we allow surreal coefficients in convex combinations.

Theorem 1 *Let A and B denote finite sets and let $(a,b) \to g(a,b)$ denote a surreal valued payoff function. If the payoff function is coherent then there exists non-negative surreal numbers p_a such that $\sum p_a = 1$ and*

$$\sum_{a\in A} p_a \cdot g(a,b) \geq 0 \tag{2}$$

for all $b \in B$.

Proof. Assume that A has d elements. Then each function $g(\cdot,b)$ may be identified with a d-dimensional surreal vector. Let K be the convex hull of $\{g(\cdot,b) \mid b \in B\}$, and let L denote the strictly negative surreal functions on A. They are convex classes.

If K and L intersect then there exists non-negative surreal numbers q_b such that $\sum q_b = 1$ and such that (1) defines a strictly negative function.

Assume that K and L are disjoint. Then define $C = K - L$ as the class of vectors $\bar{x} - \bar{y}$ where $\bar{x}$ in K and $\bar{y}$ in L. This is convex and does not contain $\bar{0}$. Now, K is a polytope (convex hull of finitely many extreme points) and L is polyhedral (given by finitely many inequalities), so C is polyhedral. Hence, each of the faces of C is given by a linear inequality of the form $\sum_{a\in A} p_a \cdot g(a) \geq c$ for $g \in C$. The delta function δ_α is non-negative so if g is in C then $g - \ell \cdot \delta_\alpha$ is also in C for ℓ positive. In particular

$$\begin{aligned} c &\leq \sum_{a\in A} p_a \cdot (g - \ell \cdot \delta_\alpha)(a) \\ &= \sum_{a\in A} p_a \cdot g(a) - \sum_{a\in A} p_a \ell \delta_\alpha(a) \\ &= \sum_{a\in A} p_a \cdot g(a) - \ell \cdot p_\alpha \end{aligned}$$

for all positive ℓ. Hence $p_\alpha \geq 0$ for all $\alpha \in A$. Further we know that $\bar{0}$ is not in C so that $\sum_{a\in A} p_a \cdot 0 \geq c$ does not hold and therefore $c > 0$. In particular p_a cannot be 0 for all a. The result follows by replacing p_a by

$$\frac{p_a}{\sum_{a\in A} p_a}.$$

■

Note that our surreal valued version Dutch Book Theorem states there are *two exclusive* cases:

1. Dutch book.
2. Non-negative mean value.

The theorem leads to surreal probabilities $p_a \geq 0$. Due to the normalization we do not have infinite probabilities, but there is no problem in having infinitesimal probabilities. In general the probability distribution will not be uniquely determined, but will merely be located in a non-empty convex set (credal set). Therefore the Dutch Book Theorem suggests that uncertainty about some unknown event should be represented by a *convex set of surreal probability distributions* rather than a single real valued distribution. Real functions are special cases of surreal functions so even if the payoff functions are real valued one can model our uncertainty by a convex set of surreal probability distributions.

If either g is acceptable or $-g$ is acceptable then it is called a two-sided bet. In this case the convex set of probability distributions reduces to a point. The term one-sided bet is taken from F. Hampel [12]. In general people will find it difficult to decide that either g or $-g$ is acceptable and thus the two-sided bet is not realistic. In De Finetti [8] only two-sided bets were considered. In our formulation of the Dutch Book Theorem we just have a one-sided bet with a set of acceptable payoff functions.

A special case that has been studied in great detail is when the functions $g(\cdot, b)$ only assume two different values, i.e. $g(\cdot, b)$ has the form

$$g(a,b) = \begin{cases} g_1(b), & \text{for } a \in A_b; \\ g_2(b), & \text{for } a \notin A_b. \end{cases}$$

Without loss of generality we may assume that $g_1(b) \geq 0 > g_2(b)$. Then the g is accepted when $P(A_b)\, g_1(b) + (1 - P(A_b))\, g_2(b) \geq 0$ or equivalently

$$P(A_b) \geq \frac{-g_2(b)}{g_1(b) - g_2(b)}. \tag{3}$$

We then define the *lower provision function* [21] by

$$L(A) = \min P(A)$$

where the minimum is taken over all distributions P that satisfies (3) for all $b \in B$. One may form surreal lower provisions in the same way as ordinary lower provisions are formed.

In this section we have seen that uncertainty may be identified with a convex set of surreal-valued probability distribution, but often such convex sets contain a lot of real-valued distributions. One may therefore ask whether the surreal-valued distributions add anything to the theory. Are they of any use? This we will try to answer in the next section.

5 Two-person zero-sum games

The theory of two persons zero sum games was founded by J. von Neumann together with O. Morgenstern [20] and has been extended to social games with more players. The readers who are interested in a deeper understanding of the theory of social games should consult [19] for an easy introduction or [10] for a more detailed exposition.

A social game with 2 players, that we will call Alice and Bob, is described by 2 sets of *strategies* A, B such that Alice can choose a strategy from A and Bob can choose a strategy from B. If Alice choose a and Bob choose b then the payoff for Alice will be $g(a,b)$ and the payoff for Bob will be $-g(a,b)$, where g is a function from $A \times B$ to surreal numbers. Alice and Bob will never collaborate in a zero-sum game because what is good for one of the players is equally bad for the other.

A pair of strategies (a,b) is called a *Nash equilibrium* if no player will benefit by changing his own strategy if the other player leaves his strategy unchanged. If a game has a unique Nash pair and both players are *rational*, then both players should play according to the Nash equilibrium.

Assume that the players are allowed to use mixed strategies, i.e. choose independent probability distributions over the strategies. The probabilities are allowed to take surreal values. Let P be the mixed strategy of Alice and Q be a mixed strategy of Bob. Then the *mean payoff* for Alice is

$$g(P,Q) = \sum_{(a,b)} g(a,b) \cdot p_a q_b.$$

This number is considered as the payoff of the social game where mixed strategies are allowed.

Theorem 2 *Consider a game with surreal valued payoffs. If the players are allowed to use mixed strategies, then the game has a Nash equilibrium.*

There exists various different proofs of the existence of Nash equilibria for two-person zero-sum games [2,10, 19,20]. In this note we shall focus on its equivalence with the Dutch Book Theorem.

The minimax inequality

$$\max_{a \in A} \min_{b \in B} g(a,b) \leq \min_{b \in B} \max_{a \in A} g(a,b)$$

is proved in exactly the same way for surreal payoff functions as for real payoff functions. The game is said to be in *equilibrium* when these quantities are equal. The common value is the *value of the game*. For any mixed strategy P for Alice the minimum of $g(P,Q)$ over distributions Q is attained when Q is concentrated in a point, i.e. $Q = \delta_b$ for some pure strategy $b \in B$. Thus

$$\min_Q g(P,Q) = \min_b \sum_a g(a,b) \cdot p_a. \tag{4}$$

To maximize this over all surreal-valued distributions P is a linear programming problem and can be solved by the same methods as if the payoff functions were real valued. In particular there exists a surreal valued distribution that maximizes (4). Using this argument we see that minimax and maximin are obtained even for mixed strategies.

Proof of equivalence of Thm. 1 and Thm. 2. Assume that for a two person zero-sum game there exists a value λ with optimal strategies P and Q. Then

g	a_1	a_2
b_1	$1+1/\omega$	$-1-2/\omega$
b_2	-1	$1+1/\omega$

Table 1: Payoff for Alice.

g	a_1	a_2
b_1	$\omega+1$	$-\omega-2$
b_2	$-\omega$	$\omega+1$

Table 2: Payoff for Alice multiplied by ω.

$\lambda < 0$ leads to the existence of a Dutch book and $\lambda \geq 0$ leads to the existence of a distribution P satisfying (2).

Assume that the Dutch Book Theorem holds. Assume that there exist a surreal number λ such that

$$\max_P \min_Q g(P,Q) < \lambda < \min_Q \max_P g(P,Q)$$

Consider the payoff function $f(a,b) = g(a,b) - \lambda$. According to the Dutch Book Theorem there exists a probability distribution P on A

$$\sum_{a\in A} p_a \cdot f(a,b) \geq 0$$

for all $b \in B$; or there exists a probability distribution Q on B such that

$$\sum_{b\in B} q_b \cdot f(a,b) < 0$$

for all $a \in A$. Therefore there exists a probability distribution P on A such that

$$\sum_{a\in A} p_a \cdot g(a,b) \leq \lambda \tag{5}$$

for all $a \in A$ or there exists a probability distribution Q on B such that

$$\sum_{b\in B} q_b \cdot g(a,b) \geq \lambda \tag{6}$$

for all strategies $a \in A$. Inequality (5) contradicts that $\lambda < \min_Q \max_P g(P,Q)$ and Inequality (6) contradicts that $\max_P \min_Q g(P,Q) < \lambda$. Hence, $\max_P \min_Q g(P,Q) = \min_Q \max_P g(P,Q)$. ■

The importance of the proof that the Dutch Book Theorem is equivalent to the existence of a Nash equilibrium for two-person zero-sum games is that it means that the two results refer to the same type of rationality. The next example show that the use of using surreal probabilities may make the difference between winning and losing.

Example 5 *Consider the payoff function in Table 1. If Alice ignores infinitesimals her optimal strategy is the distribution* $(1/2, 1/2)$, *which gives a payoff function for Bob that is* $-1/2\omega$ *if* $b = b_1$ *and* $1/2\omega$ *if* $b = b_2$. *In this case Bob could win the game by choosing* $b = b_1$. *The minimax optimal strategy for Alice is the mixed strategy* $\left(1/2 + \frac{1}{4(\omega+1)}, 1/2 - \frac{1}{4(\omega+1)}\right)$. *If she choose this mixed strategy the payoff is always positive and she will win the game.*

One should note that playing this game is not very different from playing the game where we have scaled the payoff up by a factor ω *(see Table 2). We may also scale up Bob's optimal strategy by a factor* $4(\omega+1)$ *to obtain* $(2\omega+3, 2\omega+1)$. *Therefore an optimal strategy for Alice is to play the game* $4(\omega+1)$ *"times" in parallel in such a way that* a_1 *is "chosen* $2\omega+3$ *times" and* a_2 *is "chosen* $2\omega+1$ *times ".*

If a two-persons zero-sum game has a Nash equilibrium pair $\left(\tilde{a}, \tilde{b}\right)$, which is always the case if A and B are finite, then $\sup_{a\in A} g\left(a, \tilde{b}\right) = g\left(\tilde{a}, \tilde{b}\right)$ and therefore $\inf_{b\in B} \sup_{a\in A} g(a,b) \leq g\left(\tilde{a}, \tilde{b}\right)$. Similarly, $\sup_{a\in A} \inf_{b\in B} g(a,b) \geq g\left(\tilde{a}, \tilde{b}\right)$. Thus, the game is in equilibrium and the value of the game is $g\left(\tilde{a}, \tilde{b}\right)$. In particular all Nash equilibria have the same value. The same argument holds for mixed strategies.

6 Dutch books for short games

Surreal numbers are totally ordered and never confused with each other. Games that are not surreal number are confused with a small or large interval of surreal numbers. For instance $*$ is confused with 0 and the game $\{100 \mid -100\}$ is confused with any number between -100 and 100. Before formulating a Dutch Book Theorem for general combinatorial games we need to introduce the *mean value* $\mu(G)$ of a short game G. A game G is said to be *short* if it only has finitely many positions. Our recursive definition of games allows transfinite recursion and games that are not short, but for the definition of mean values we shall focus on the short games. Note that if a short game is a number then it is a dyadic fraction.

The mean value of a game G is a real number $\mu(G)$ that satisfies the following mean value theorem.

Theorem 3 ([7]) *If* G *is a short game then there exists a natural number* m *and a number* $\mu(G)$ *that satisfies*

$$n \cdot \mu(G) - m \leq n \cdot G \leq n \cdot \mu(G) + m$$

for all natural numbers n.

Mean values of short games can be calculated by the *thermographic method* described in [7] and using this method it is easy to see that the mean value of a short game is always a rational number. Mean values of games share some important properties with mean values of random variables. For instance we have

- $\mu(n \cdot G) = n \cdot \mu(G)$,
- $\mu(G + H) = \mu(G) + \mu(H)$,
- $G \geq 0 \Rightarrow \mu(G) \geq 0$,
- $\mu(1) = 1$.

Example 6 *The game $G = \{1 \mid \{0 \mid -2\}\}$ that is illustrated in Figure 2, satisfies $G > 0$. In the game $n \cdot G$ Right can only play in a sub-game where Left has not played and the response optimal for Left is always to answer a move of Right by a move in the same sub-game. From this one sees that $n \cdot G \leq 1$ and therefore that $\mu(G) = 0$. We see that Left may win a game* for sure *although the game has mean value zero!*

The setup is as before that each bookmaker $b \in B$ tells Alice which game he wants to play if a certain horse $a \in A$ wins. Alice is going to play Left and the bookmaker or the bookmakers are going to play Right. After certain bookmakers have been accepted the bookmakers choose natural numbers $n_b, b \in B$ and combine these into a super game $\sum_{b \in B} n_b \cdot G(a, b)$ that will depend on which horse wins. We say that we have a *Dutch book* if there exists natural numbers $n_1, n_2, \cdots, n_k$ such that Alice will lose the game

$$\sum_{b \in B} n_b \cdot G(a, b) \tag{7}$$

for any value of a. Otherwise the set of game valued payoff functions is said to be *coherent*. If all the games are short surreal numbers then this notion of coherence is equivalent to the definition of coherence given in Section 4.

Alice is allowed to choose that the game should be played a number of times in parallel. With this setup we get the following version of the Dutch Book Theorem.

Theorem 4 *If a payoff function $G(a, b), a \in A, b \in B$ with short games as values, is coherent then either exists a probability vector $a \to p_a$ and a natural number n such that $np_a \in \mathbb{N}$ and the game*

$$\sum_a (np_a) \cdot G(a, b) > 0, \text{ for all } b \in B, \tag{8}$$

or there exist natural numbers $n_1, n_2, \cdots, n_k$, a natural number n and a probability vector $a \to p_a$ such that both games (7) and (8) have mean value 0.

Proof. We apply the existence of an equilibrium in the two-person zero-sum game with payoff function $(a, b) \to \mu(G(a, b))$. If the value of the two-person zero-sum game is negative then the game (7) is negative if the coefficients $n_1, n_2, \cdots, n_k$ are large enough. If the value of the two-person zero-sum game is non-negative there exists a probability vector $a \to p_a$ such that

$$\sum_a p_a \cdot \mu(G(a, b)) \geq 0.$$

The mean value of a short game is a rational number. Therefore the probability vector $a \to p_a$ can be chosen with rational point probabilities. Hence, there exists a natural number m such that $m \cdot p_a$ is an integer for all $a \in A$. Therefore

$$\begin{aligned} 0 &\leq m \sum_a p_a \cdot \mu(G(a, b)) \\ &\leq \sum_a mp_a \cdot \mu(G(a, b)) \\ &= \mu\left(\sum_a mp_a \cdot G(a, b)\right). \end{aligned}$$

If

$$\mu\left(\sum_a mp_a \cdot G(a, b)\right) > 0$$

then there exists a natural number k such that

$$k \sum_a mp_a \cdot G(a, b) > 0$$

and the game defined in (8) is winning for Alice who plays as Left when $n = km$. Otherwise

$$\mu\left(\sum_a mp_a \cdot G(a, b)\right) = 0. \tag{9}$$

■

Here we should note that our short-game-valued Dutch Book Theorem stated there are *three* cases that are *not exclusive:*

1. Dutch book.
2. Positive mean.
3. Zero mean.

As we saw in Example 6 a game with mean zero may be positive or negative. Therefore a decision strategy in which only games with positive means are acceptable will exclude some games that one will win for sure and a decision strategy where games with non-negative mean are acceptable will include some games that are lost for sure. The most reasonable solution to this problem seems to be to accept or reject according to the mean payoff with respect to some probability distribution, but leave the cases with mean zero undecided because a more detailed non-probabilistic analysis is needed for these cases.

7 More on infinitesimals

The Dutch Book Theorem for short games only used rational valued mean values. One may hope for a better Dutch Book Theorem if we allow also allow a mean value function with infinitesimal surreal numbers as mean values. For short games this will not solve the problem.

Definition 2 *A game G is said to be* strongly infinitesimal *if $-s \leq G \leq s$ for any surreal number $s > 0$.*

Example 7 *The game $\{0 \mid *\}$ is called* up *and denoted $\uparrow$. It is easy to check that $\uparrow > 0$. The game $\uparrow$ is infinitesimal (check how Left can win $2^{-s} - \uparrow$). One can prove that any infinitesimal short game is strongly infinitesimal [18].*

An interesting situation is when all games $G(a, b)$ are infinitesimal. In this case the Dutch Book Theorem for games as formulated in Theorem 4 tells exactly nothing because the mean value of strongly infinitesimal games would always be 0 even if surreal mean values are allowed. But if all games are infinitesimal one could shift to a different "mean value" concept. For short games one compares the game with $n \cdot 1$ and the game 1 can be considered as a unit in the theory. For infinitesimal short games one can compare with the infinitesimal game $\uparrow$ instead. It is possible to define an *atomic mean value* such that $\uparrow$ has mean 1, but the proofs are more involved. One can also prove a version of the Dutch Book Theorem for infinitesimally short games that involves three cases. The three cases are Dutch book, positive mean, and some games G that cannot be analyzed in the sense that their atomic mean value is zero. Although infinitesimal games can be treated with their own mean value concept this will not solve all problems because games that are not infinitesimal may sometimes be combined into strongly infinitesimal games. A simple example consist of the games 1 and $\uparrow - 1$ whose sum is the strongly infinitesimal game $\uparrow$.

8 Discussion

In any frequency interpretation of probability theory, probabilities should be interpreted as limits of frequencies. Obviously surreal probabilities cannot have such interpretations because a frequency interpretation cannot distinguish between surreal probabilities that have an infinitesimal difference. This leads us to the following conclusion: frequency probabilities are real numbers but uncertainty should in general be modelled by convex sets of surreal numbers.

In a *subjective Bayesian* approach to probability and statistics one will assign probabilities expressing the individual feeling of how probable or likely an event is. Many subjective Bayesians justify this point of view by reference to the Dutch Book Theorem. We note that unlike some of the modification by Savage et al. neither our formulation of the Dutch Book Theorem nor its original formulation of de Finetti has any reference to subjectivity. For short-game valued payoffs even the one-to-one correspondence between probability and coherent decisions breaks down. Experiments have demonstrated that most people have a bad intuition of probabilities and are unable to assign probabilities in a consistent manner. It should be even harder to make a consistent distinction between the probabilities $1/3$ and $1/3 + 1/\omega$ although the Dutch Book Theorem give the same type of justification for surreal probabilities as for real probabilities.

We have seen that from a mathematical point of view uncertainties may be modeled by a convex set of surreal probability vectors, but the reader may wonder why infinitesimals do normally not appear in probability theory. Actually there are many real numbers that never appear as probabilities. For instance all the numbers that *do* appear are *computable*, and there are only countably many computable numbers. Therefore, it seems that the use of surreal numbers is an idealization that is not worse than the use of real numbers as subjective probabilities. At the moment two-person zero-sum games like the ones described in Example 5 are the only known kind of calculations that gives surreal valued probabilities as results.

In this paper we used the operations + and $\cdot$ to define Dutch books and coherence. These operations refer to ways of combining games into new games. It is an open question what kind of Dutch Book Theorem one would get if other ways of combining games were considered.

For social games with several players and surreal-valued payoff functions we have not been able to prove existence of a Nash equilibrium, because one cannot use the usual fixed-point results that rely heavily on the topology of the real numbers. We shall not discuss it here as it has less interest for our understanding of what probabilities are.

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