On the use of a new discrepancy measure to correct incoherent assessments and to aggregate conflicting opinions based on imprecise conditional probabilities

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Abstract

We give a preliminary study of a new procedure to correct incoherent imprecise conditional probability assessments. The procedure is based on parametric optimization problems which have as objective function a new discrepancy measure. We show through simple examples how the procedure of correcting incoherent assessments can be properly extended to aggregate conflicting opinions, and can be generalized to embed importance weights of each assessment.

Keywords. Imprecise conditional probabilities, inconsistency handling, aggregation opinions, divergence measures.

1 Introduction

In this paper we illustrate a preliminary study for the adoption of a new procedure to correct inconsistent imprecise conditional probability assessments. The procedure is based on parametric optimization problems the objective function of which is a discrepancy measure recently introduced in [4] for a similar purpose with respect to precise assessments. Such discrepancy originates from a peculiar choice of a scoring rule, and it behaves like ordinary divergences among probability distributions.

Care must be taken for the notion of incoherence. In fact, for imprecise conditional probability assessments, different coherence requirements are possible (see e.g. the comparison among them done in [20, 21]). We choose to proceed along the line of de Finetti [12, 13], adopting the most stringent generalization of his coherence notion for precise assessments to imprecise ones, as proposed by Coletti and Scozzafava (see e.g. [7]).

Assessments inconsistency can naturally arise whenever there is the need to merge different sources of uncertainty information. The extension of our correction procedure to aggregation of opinions comes quite naturally. It is in fact sufficient to formally duplicate the elements in common among the assessments to have a joint one, and treat it as generated by a unique source.

Aggregation of different opinions is actually a subject which has been studied in depth, both in precise (see e.g. [10, 14, 22, 25]) and imprecise (see e.g. [11, 16, 19, 23, 24]) evaluation frameworks. Some aggregation rules are based solely on the assessed values, others rely on auxiliary over structures, like for example second order assessments or risk neutral probabilities. Our choice lies in between: once a specific scoring rule is chosen, the aggregation proceeds “alone” by working only on the assessed values.

The procedure we propose reveals its efficacy especially whenever opinions are given on different domains and the envelope of opinions union turns out to be incoherent “per se”.

While theoretical details will be the object of a future contribution, we present here some simple examples to show peculiarities and potentialities of our procedure.

The paper continues with Section 2, where the notation and basic notions are introduced. In particular, the discrepancy mentioned above is described and its justification and properties are reported. After that, in Section 3 we illustrate how to use such discrepancy as objective function of parametric optimization problems, so that, by an iteration, it is possible to select a set of coherent precise assessments whose lower-upper envelopes induce the correction of an initially incoherent assessment. In Section 4 we extend the procedure to the aim of aggregating different opinions. This generalization comes quite naturally by a simple rewriting of the joint assessment. After that we generalize the discrepancy measure by introducing a weighted version. In fact, it is possible to differentiate the importance of the single opinions, and inside them of the single assessed values. Finally, we end by Section 6, where a short conclusion is reported.

2 Basic notions

We formalize the domain of the evaluation through a finite family of conditional events of the type $\mathcal{E} = [E_1|H_1, \ldots, E_n|H_n]$.

Events E_i-s usually represent the situations under consideration, while the H_i-s usually represent the different contexts, or scenarios, under which the E_i-s are evaluated.

The basic events $E_1, \ldots, E_n, H_1, \ldots, H_n$ can be endowed with logical constraints, that represent dependencies among particular configurations of them (e.g. incompatibilities, implications, partial or total coincidences, etc.).

In the following E_iH_i will denote the logical connection "E_i and H_i", $\neg E_i$ will indicate "not E_i" and the event $H^0 = \bigvee_{i=1}^n H_i$ will represent the whole set of contexts considered.

By the basic events $E_1, \ldots, E_n, H_1, \ldots, H_n$ it is possible to span a sample space $\Omega = \{\omega_1, \ldots, \omega_k\}$, where ω_j represents generic atoms, in some context named "possible worlds ". Note that the sample space Ω and H^0 are not part of the assessment but only auxiliary tools.

The numerical part of the assessment is elicited through interval values

$$\mathbf{lub} = ([lb_1, ub_1], \ldots, [lb_n, ub_n]) \qquad (1)$$

thought as honest ranges for the probabilities $p_i = P(E_i|H_i)$, $i = 1, \ldots, n$. Of course, some of the intervals $[lb_i, ub_i]$-s could degenerate to precise values p_i-s.

For assessments like $(\mathcal{E}, \mathbf{lub})$, although defined on finite spaces, there could be different kinds of consistency requirements (for a detailed exposition, among others, refer to [20]). In this paper we focus on the most stringent one: (strong) coherence. By adopting a Bayesian sensitivity analysis interpretation, coherent lower-upper conditional probability assessments $(\mathcal{E}, \mathbf{lub})$ are those the numerical part $\mathbf{lub}$ of which can be obtained as lower-upper envelopes of sets of coherent precise, i.e. linear, conditional probability assessments on $\mathcal{E}$; coherence for precise assessments is thought in the most general sense of restrictions on $\mathcal{E}$ of full finitely additive conditional probability distributions. For a complete and rigorous description see the exhaustive treatise [9].

It follows that to have a coherent assessment on $\mathcal{E}$, there should exist a set of probability distributions over Ω such that, on one hand it induces probabilities for the $E_i|H_i$-s inside the ranges $[lb_i, ub_i]$, and on the other hand it is such that each lower (lbi_s) and upper (ubi_s) bound of the ranges is attained through at least one distribution in the set.

We denote by $\mathcal{M}$ such set of coherent precise conditional assessments compatible with $(\mathcal{E}, \mathbf{lub})$

$$\begin{aligned} \mathcal{M} \;\; := \;\; & \{P \text{ coherent} \,|\, lb_i \leq P(E_i|H_i) \leq ub_i, \\ & i = 1, \ldots, n\}. \end{aligned} \qquad (2)$$

We shall focus on the situations with an empty $\mathcal{M}$ that characterize incoherent assessments $(\mathcal{E}, \mathbf{lub})$. Such kind of incoherence is usually denoted as "incurring in uniform loss" (see [27]) or as "not g-coherent" (see [1]).

In literature it is commonly faced an other kind of incoherence: $\mathcal{M}$ is not empty but there exist at least one index $i \in \{1, \ldots, n\}$ such that

$$lb_i < \inf_{P \in \mathcal{M}} P(E_i|H_i) \quad \text{or} \quad \sup_{P \in \mathcal{M}} P(E_i|H_i) < ub_i \quad . \qquad (3)$$

In this cases $(\mathcal{E}, \mathbf{lub})$ is said to "avoid uniform loss but not strong coherent" (see [26, 28]), or simply "incoherent" (see [7]). This second kind of incoherence can be directly solved by computing the "natural extension" of $(\mathcal{E}, \mathbf{lub})$ (see again [1, 21, 27], among others).

Actually, there is a third type of incoherence: when the assessment is not "weak coherent" (see again [26]). For finite domains, this subtle "weak incoherence" derives, as well illustrated in [20, 21], by the exclusion of conditioning on events with zero probability. Since, on the contrary, we believe that it is important to include such assessments in $\mathcal{M}$ (see e.g. [8]), we do not tackle this further type of inconsistency.

Whenever $(\mathcal{E}, \mathbf{lub})$ incurs in a uniform loss, there is no unique way to adjust it. In this paper we propose to find out the "closest" correction, with a specific choice for the "distance" notion. In [4] we already did this for precise assessments taking advantage of the aforementioned discrepancy measure. We propose now to extend such method to imprecise assessments by generalizing the discrepancy among sets of assessments.

Before introducing the discrepancy measure, we need some further auxiliary notions.

Every probability distribution $\alpha : \mathcal{P}(\Omega) \to \mathbb{R}$ corresponds to a non-negative vector $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_k]$, with $\alpha_j = \alpha(\omega_j)$; then for every event E it will be $\alpha(E) = \sum_{\omega_j \subseteq E} \alpha_j$. We will refer to a nested hierarchy of probability distributions over Ω. This to properly separate inner from boundary situations:

- let $\mathcal{A} := \{\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_k] \,|\, \sum_1^k \alpha_i = 1, \alpha_j \geq 0, j = 1, \ldots, k\}$ represents the whole set of probability distributions on Ω;

- let $\mathcal{A}_0 := \{\boldsymbol{\alpha} \in \mathcal{A} | \alpha(H^0) = \alpha(\bigvee H_i) = 1\}$ be the subset of probability distributions on Ω that concentrate all the probability mass on the contemplated scenarios[1];
- let $\mathcal{A}_1 := \{\boldsymbol{\alpha} \in \mathcal{A}_0 | \alpha(H_i) = \sum_{\omega_j \in H_i} \alpha_j > 0, i = 1, \ldots, n\}$ be the subset of probability distributions on Ω that give positive probability to every scenario;
- let $\mathcal{A}_2 = \{\boldsymbol{\alpha} \in \mathcal{A}_1 | 0 < \dfrac{\sum_{j:\, \omega_j \subset E_i H_i} \alpha_j}{\sum_{j:\, \omega_j \subset H_i} \alpha_j} < 1,$ $i = 1, \ldots, n\}$ be the subset of probability distributions that avoid boundary values $\{0, 1\}$ for the conditional probabilities.

Any $\boldsymbol{\alpha} \in \mathcal{A}_1$ induces a coherent precise conditional assessment on $\mathcal{E}$

$$\mathbf{q}_{\boldsymbol{\alpha}} := [q_i = \frac{\sum_{j:\, \omega_j \subset E_i H_i} \alpha_j}{\sum_{j:\, \omega_j \subset H_i} \alpha_j}, i = 1, \ldots, n]. \qquad (4)$$

Associated to any (coherent or not) precise assessment $\mathbf{p} = [p_1, \ldots, p_n] \in (0,1)^n$ over $\mathcal{E} = [E_1|H_1, \ldots, E_n|H_n]$ we can introduce a scoring rule

$$S(\mathbf{p}) := \sum_{i=1}^{n} |E_i H_i| \ln p_i + \sum_{i=1}^{n} |\neg E_i H_i| \ln(1 - p_i) \quad (5)$$

with $|\cdot|$ indicator function of unconditional events.

Note that such scoring rule is not defined for boundary values 0 or 1 of the assessed probabilities. This is of course a limitation of our approach, but all the same we believe it is significant. In fact if the assessor had so strong a belief in assessing such extreme values, it could mean that the component did not want to be the object of a settlement. Hence if any of the lbi_s or of the ubi_s turn out to be 0 or 1, they are maintained fixed in their values, if this of course will not induce any evident contradiction, otherwise they must be treated outside our procedure.

This score $S(\mathbf{p})$ is an "adaptation" of the "proper scoring rule" for probability distributions proposed by Lad in [18](pag. 355). We have extended it to partial and conditional probability assessments.

Such a score is motivated by a conditional event $E_i|H_i$ being a three-valued logical entity, partitioning Ω in three parts (omnia Gallia divisa est in partes tres): the atoms satisfying $E_i H_i$ and therefore verifying the conditional, those satisfying $\neg E_i H_i$, thus falsifying the conditional, and those not fulfilling the context H_i, to which the conditional may not be applied at all. Hence the assessor of $\mathbf{p}$ "loses less" the higher are the probabilities assessed for events that are verified, and at the same time, the lower are the probabilities assessed for those that are not verified. The values assessed on events that turn out to be undetermined do not influence the score. In fact the realization of the random value $S(\mathbf{p})$ when the atom ω_j occurs is

$$S_j(\mathbf{p}) = \sum_{i:\, E_i H_i \supset \omega_j} \ln p_i + \sum_{i:\, \neg E_i H_i \supset \omega_j} \ln(1 - p_i). \quad (6)$$

The simultaneous involvement in this score of events that turn out to be true and of those that turn out to be false modifies the peculiar property of the usual logarithmic scoring rule to depend only on the true ones.

We now have all the elements to introduce the "discrepancy" between a precise assessment $\mathbf{p}$ over $\mathcal{E}$ and a distribution $\boldsymbol{\alpha} \in \mathcal{A}_2$, with respect to its induced conditional coherent assessment $\mathbf{q}_{\boldsymbol{\alpha}}$, as

$$\begin{aligned} \Delta(\mathbf{p}, \boldsymbol{\alpha}) &:= E_{\boldsymbol{\alpha}}(S(\mathbf{q}_{\boldsymbol{\alpha}}) - S(\mathbf{p})) & (7) \\ &= \sum_{j=1}^{k} \alpha_j [S_j(\mathbf{q}_{\boldsymbol{\alpha}}) - S_j(\mathbf{p})]. & (8) \end{aligned}$$

The distributions $\boldsymbol{\alpha}$ are restricted to be in $\mathcal{A}_2$ because only there the scoring rule $S(\mathbf{q}_{\boldsymbol{\alpha}})$ is properly defined. It is however possible to extend by continuity the previous definition of $\Delta(\mathbf{p}, \boldsymbol{\alpha})$ to any distribution $\boldsymbol{\alpha}$ in $\mathcal{A}_0$ through the expression

$$\Delta(\mathbf{p}, \boldsymbol{\alpha}) = \sum_{i=1}^{n} \ln(\frac{q_i}{p_i}) \alpha(E_i H_i) + \ln(\frac{1 - q_i}{1 - p_i}) \alpha(\neg E_i H_i) \quad (9)$$

$$= \sum_{i=1}^{n} \alpha(H_i) \left(q_i \ln(\frac{q_i}{p_i}) + (1 - q_i) \ln(\frac{1 - q_i}{1 - p_i}) \right). \quad (10)$$

This discrepancy $\Delta(\mathbf{p}, \boldsymbol{\alpha})$ behaves in a way that is analogous to other usual Bregman divergences[2] (see [2]). In fact in[5] we formally proved that the following properties hold:

- $\Delta(\mathbf{p}, \boldsymbol{\alpha}) \geq 0 \quad \forall \boldsymbol{\alpha} \in \mathcal{A}$;
- $\Delta(\mathbf{p}, \boldsymbol{\alpha}) = 0$ iff $\mathbf{p} \equiv \mathbf{q}_{\boldsymbol{\alpha}}$;
- $\Delta(\mathbf{p}, \cdot)$ is convex on $\mathcal{A}_2$;
- $\Delta(\mathbf{p}, \cdot)$ always admits a minimum on $\mathcal{A}_0$;

[1] This is commonly done in conditional frameworks to avoid unpleasant consequences. See Walley[26] about Avoiding Uniform Loss assessments or Holzer[17] about the Principle of Conditional Coherence

[2] Actually $\Delta(\mathbf{p}, \boldsymbol{\alpha})$ turns out to be a generalization of the sum of two different "Bregman divergences".

- If $\Delta(\mathbf{p}, \cdot)$ attains its minimum value on $\mathcal{A}_1$; then there is a unique coherent assessment $\mathbf{q}_{\underline{\alpha}}$ on $\mathcal{E}$ such that $\Delta(\mathbf{p}, \underline{\boldsymbol{\alpha}})$ is minimum;
- If $\Delta(\mathbf{p}, \cdot)$ attains its minimum value on $\mathcal{A}_0 \setminus \mathcal{A}_1$, then any distribution $\boldsymbol{\alpha} \in \mathcal{A}_0$ that minimizes $\Delta(\mathbf{p}, \cdot)$ induces the same significant conditional probabilities $(\mathbf{q}_\alpha)_j$ on the conditional events $E_j|H_j$ such that $\alpha(H_j) > 0$.

The last two items are the crucial ones: for precise numerical evaluations $\mathbf{p}$, they always guarantee the existence of a coherent assessment $(\mathcal{E}, \mathbf{q}_{\widetilde{\boldsymbol{\alpha}}})$ "close as much as possible" to $(\mathcal{E}, \mathbf{p})$. And this also with respect to the most general notion of conditional coherence that contemplates the hierarchy of the so called "zero layers" (see again [9] for details about this delicate and crucial notion).

3 Correcting incoherent assessments

Let us see how the properties of $\Delta(\mathbf{p}, \boldsymbol{\alpha})$ could help us in the correction of an incoherent assessment.

The starting point is that incoherence of $(\mathcal{E}, \mathbf{lub})$ is equivalent to the incoherence of any precise assessment $\mathbf{v} = (v_1, \ldots, v_n)$ with $lb_i \leq v_i \leq ub_i$. On the other hand, the assessor elicitates the bounds lb_i-s and ub_i-s as effectively attainable. For this reason, we iteratively fix a specific bound lb_f (or ub_f) and we find the precise coherent assessment $\tilde{\mathbf{q}}$ that is the closest to the subset of precise assessments $\mathbf{v}$-s that reach lb_f (or ub_f), while remaining inside the ranges $[lb_i, ub_i]$-s for the others elements.

More precisely, by fixing an index $f \in \{1, \ldots, n\}$, we can find two coherent assessments $\underline{\mathbf{q}}_f$ and $\overline{\mathbf{q}}_f$ on $\mathcal{E}$, induced respectively, by the solutions of the following two parametric optimization problems, with parameter $\mathbf{v}$:

$$\text{minimize } \Delta(\mathbf{v}, \boldsymbol{\alpha}) \tag{11}$$

under the constraints

$$v_f = lb_f \quad \textbf{or} \quad v_f = ub_f \tag{12}$$

$$\forall i \neq f \quad lb_i \leq v_i \leq ub_i \quad , \quad i \in \{1, \ldots, n\} \tag{13}$$

$$\sum_{j:\, \omega_j \subset E_k H_k} \alpha_j = q_k \sum_{j:\, \omega_j \subset H_k} \alpha_j \quad , \quad k = 1, \ldots, n \tag{14}$$

$$\boldsymbol{\alpha} \in \mathcal{A}_0 \quad . \tag{15}$$

The choice in (12) of whether to fix the lower or upper bound distinguishes one problem from the other.

The $n-1$ constraints (13) reflect the compatibility of $\mathbf{v}$ with the other intervals in $\mathbf{lub}$, while the n constraints (14) impose the coherence of the assessment $\mathbf{q}_{\boldsymbol{\alpha}}$ induced by a solution $\boldsymbol{\alpha}$.

Note that if any optimal solution $\widetilde{\boldsymbol{\alpha}}$ of (11) is in $\mathcal{A}_0 \setminus \mathcal{A}_1$, the associated conditional assessment $\mathbf{q}_{\widetilde{\boldsymbol{\alpha}}}$ is properly defined only for those conditional events $E_k|H_k$ with $\widetilde{\alpha}(H_k) > 0$, some component of $\underline{\mathbf{q}}_f$ (or of $\overline{\mathbf{q}}_f$) remaining unspecified. Hence, in these cases, we need to explore other "zero layers". This can be simply done by reiterating the optimization problem over the part of $\mathcal{E}$ with probability of the conditioning events induced by $\widetilde{\boldsymbol{\alpha}}$ equal to 0. The new optimal solutions are distributions defined on sample spaces spanned by the sub-domain, so that they significantly induce some of the unspecified component of $\underline{\mathbf{q}}_f$ (or $\overline{\mathbf{q}}_f$). Since for each iteration there will be at least one conditioning event H_k with strictly positive induced probability, at worst in $n-1$ steps the assessments $\underline{\mathbf{q}}_f$ (or $\overline{\mathbf{q}}_f$) are fully determined.

By letting the index f vary over the full range $1, \ldots, n$ we obtain a set of $2n$ coherent assessments

$$\mathcal{Q} = \{\underline{\mathbf{q}}_f, \overline{\mathbf{q}}_f, f = 1, \ldots, n\}. \tag{16}$$

By definition, the imprecise assessment on $\mathcal{E}$

$$\mathbf{luc} = ([lc_1, uc_1], \ldots, [lc_n, uc_n]), \tag{17}$$

which is bounded by the lower and upper envelope of $\mathcal{Q}$, i.e.

$$lc_i := \min_{\tilde{\mathbf{q}} \in \mathcal{Q}} \tilde{\mathbf{q}}(E_i|H_i) \qquad uc_i := \max_{\tilde{\mathbf{q}} \in \mathcal{Q}} \tilde{\mathbf{q}}(E_i|H_i), \tag{18}$$

is coherent and can be adopted as correction of $\mathbf{lub}$.

Note moreover that we have no guarantees about the uniqueness of $\underline{\mathbf{q}}_f$, or of $\overline{\mathbf{q}}_f$, because the set of optimal solutions

$$\mathcal{O}_f = \{\widetilde{\boldsymbol{\alpha}} \in \mathcal{A}_0 | \widetilde{\boldsymbol{\alpha}} \text{ optimal solution of } (11-15)\} \tag{19}$$

could induce different coherent precise assessments over $\mathcal{E}$. At the moment, numerical experiments support uniqueness, but further theoretical investigations are needed. In any case, if there were different assessments induced by (19), we could take the whole set of them instead of the single $\underline{\mathbf{q}}_f$ (or $\overline{\mathbf{q}}_f$) to determine the envelope (18).

Let us see how our correction procedure works with a simple example.

Example 1 *By borrowing the framework from [15], we consider the domain $\mathcal{E} = [C|A, C|B, C|A \vee B]$ built by three basic unconditional logically independent events A, B, C. Hence the whole sample space would be of 8 atoms, but only 6 are inside $H^0 \equiv A \vee B$. The set of coherent assessments on $\mathcal{E}$ is made by the triples*

$[q_1, q_2, q_3] \in [0,1]^3$, *with the last component forced to be in the range*

$$q_3 \in \left[\frac{q_1\, q_2}{q_1 + q_2 - q_1\, q_2}, \frac{q_1 + q_2 - 2q_1\, q_2}{1 - q_1\, q_2}\right] \qquad (20)$$

(see Fig.1). Note the evident non-convexity of this coherent set.

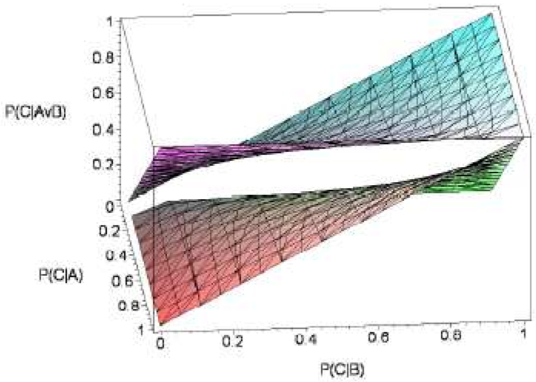

Figure 1: Lower and upper bounds for coherent assessments on $\mathcal{E} = [C|A, C|B, C|A \vee B]$

Let us firstly consider an assessments **lub** *that incurs in a uniform loss*

$\mathcal{E}$	$C\|A$	$C\|B$	$C\|A \vee B$
lb_i	.1	.2	.6
ub_i	.3	.4	.8

. (21)

Incoherence can be highlighted by taking as good the first two components of **lub**, *so that the coherent (natural) extension to $C|A \vee B$ should be, by (20), the interval* $[.0714, .5227]$, *that does not overlap the assessed range* $[.6, .8]$. *Hence we have that the set $\mathcal{M}$ of precise assessments compatible with* **lub** *is empty.*
By performing [3] *the 6 optimization problems of type (11), we obtain the following coherent precise assessments:*

$\mathcal{E}$	$C\|A$	$C\|B$	$C\|A \vee B$
$\underline{\mathbf{q}}_1$	.1196	.4797	.5140
$\overline{\mathbf{q}}_1$	.3263	.4344	.5560
$\underline{\mathbf{q}}_2$	.3830	.2558	.4910
$\overline{\mathbf{q}}_2$	.3263	.4344	.5560
$\underline{\mathbf{q}}_3$	.3263	.4344	.5560
$\overline{\mathbf{q}}_3$	.4078	.5440	.6530

, (22)

whose lower-upper envelope results the following coherent imprecise assessment:

$\mathcal{E}$	$C\|A$	$C\|B$	$C\|A \vee B$
lc_i	.1196	.2558	.4910
uc_i	.4078	.5440	.6530

. (23)

[3] Numerical results obtained with the nonlinear optimization software CONOPT of the GAMS package [3]

4 Aggregating conflicting opinions

The merging, or aggregation, of different opinions has a considerable importance for both theoretical and practical aspects. This subject has been widely treated in several scientific fields, and even restricting attention to probabilistic models, there is a great number of proposals. An interesting feature occurs when the different opinions are in conflict, i.e. the whole assessment results incoherent (see e.g. [6, 25] for precise assessments and [16, 19] for imprecise ones).

In our approach, conflict among opinions can be expressed through disjoint intervals associated to the same conditional events, and/or through incoherence of the joint assessment.

Here we propose to adopt the previous procedure, which we have seen to correct incoherent imprecise assessments, also for aggregation purposes.

First of all, if we have evaluations assessed on $\mathcal{E}^s = [E_{1.s}|H_{1.s}, \ldots, E_{n.s}|H_{n.s}]$, with the index $s \in S$ expressing the different sources, we denote the joint domain by $\mathcal{E} = \bigvee_{s \in S} \mathcal{E}^s$.

Secondly, we can replace the possible multiple ranges assigned to single elements of $\mathcal{E}$ duplicating such elements and adding coincidence constraints in list of the logical relationships. For example, if we have two different ranges $[lb_i', ub_i']$ and $[lb_i'', ub_i'']$ associated to the same $E_i|H_i \in \mathcal{E}$, we can actually associate the second interval $[lb_i'', ub_i'']$ to a new conditional event $E_i''|H_i''$ added to $\mathcal{E}$, and increase the logical relationships with the constraints

$$E_i H_i \equiv E_i'' H_i'' \quad ; \qquad (24)$$

$$H_i \equiv H_i'' \quad . \qquad (25)$$

In this way, we will have the different opinions joined in a single imprecise (and incoherent) assessment of the type $(\mathcal{E}, \mathbf{lub})$, so that its correction $(\mathcal{E}, \mathbf{luc})$ will represent an aggregation result. Of course, since **luc** is a coherent imprecise assessment, equal intervals ($[lc_i', uc_i'] = [lc_i'', uc_i'']$) will be associated to coincident elements of $\mathcal{E}$ ($E_i|H_i$ and $E_i''|H_i''$).

Let us see how this works with an example.

Example 2 *Let us consider again the framework of the previous Example 1, but now with two different opinions given on separate, but overlapping, subdomains:*

	$C\|A$	$C\|B$	$C\|A \vee B$
$\mathbf{lub}'$	$[.1, .3]$	$[.2, .4]$	–
$\mathbf{lub}''$	–	$[.5, .7]$	$[.6, .8]$

, (26)

by duplicating $C|B$, we obtain a unique whole assessment:

$\mathcal{E}$	$C\|A$	$C\|B$	$C''\|B''$	$C\|A \vee B$
lub	$[.1, .3]$	$[.2, .4]$	$[.5, .7]$	$[.6, .8]$

(27)

with the logical constraints

$$C''B'' \equiv CB \quad , \quad B'' \equiv B. \tag{28}$$

The 8 iterations of the optimization problem of type (11) give the following set $\mathcal{Q}$ of coherent precise assessments:

$\mathcal{E}$	$C\|A$	$C\|B$	$C''\|B''$	$C\|A \vee B$
$\underline{\mathbf{q}}_1$	.1193	.4872	.4872	.5205
$\overline{\mathbf{q}}_1$	.3196	.4612	.4612	.5700
$\underline{\mathbf{q}}_2$	.4053	.3678	.3678	.4855
$\overline{\mathbf{q}}_2$	.3196	.4612	.4612	.5700
$\underline{\mathbf{q}}_3$	.3196	.4612	.4612	.5700
$\overline{\mathbf{q}}_3$	.3547	.5749	.5749	.5380
$\underline{\mathbf{q}}_4$	.3196	.4612	.4612	.5700
$\overline{\mathbf{q}}_4$	.4078	.5440	.5440	.6530

(29)

lower-upper envelope of which gives us the coherent aggregation

$\mathcal{E}$	$C\|A$	$C\|B$	$C\|A \vee B$
luc	$[.1193, .4078]$	$[.3678, .5749]$	$[.4855, .6530]$

. (30)

Of course, the approach doesn't change if more than two assessments are given to the same conditional event $E_i|H_i$. We simply have as many coincidence constraints (24,25) as assessed intervals for $E_i|H_i$.

Note that the aggregation (30) we obtained in the previous example deforms all the original opinions (26). This is because the two assessments are strongly in conflict. In fact, apart from the obvious inconsistence due to the two disjoint intervals given on $C|B$, the range $[.6, .8]$, in $\mathbf{lub}''$ associated to $C|A \vee B$, does not overlap the natural extension of $\mathbf{lub}'$

$$[lb'_{C|A\vee B}, ub'_{C|A\vee B}] = [.0714, .5227] \quad . \tag{31}$$

But there are cases in which our procedure gives an aggregation result that reconciles, without misshaping, the original assessments. We can see this in the next example.

Example 3 *If we modify the two separate opinions (26) of the previous example to*

	$C\|A$	$C\|B$	$C\|A \vee B$
$\mathbf{lub}'$	$[.1, .3]$	$[.35, .6]$	$-$
$\mathbf{lub}''$	$-$	$[.3, .55]$	$[.1, .6]$

, (32)

our procedure (we skip here the detailed computations of $\mathcal{Q}$) gives as lower-upper envelope the assessment

$\mathcal{E}$	$C\|A$	$C\|B$	$C\|A \vee B$
luc	$[.1, .3]$	$[.3, .6]$	$[.1, .6]$

(33)

that coincide with the least commitment aggregation of (32), that is the lower-upper envelope of the union of the single intervals.

More generally, we can emphasize that our aggregation procedure is particularly significant when joining the different opinions gives an incoherent result, so that each assessed interval influences the result of the merging. On the other hand, note that if all the original opinions are coherent and given on the same domain $\mathcal{E}$, our aggregation result coincides with the least commitment aggregation mentioned above, also named "unanimity rule". Although a "weak" result, this coincidence allows us to compare the behavior of our procedure with many properties for aggregation rules suggested by various authors (see for example [10, 14, 19, 24] among others). In fact, in such situations we trivially have that

- *Unanimity Preservation*: if all the experts agree and give the same assessments for the same events, then the aggregate agrees with all the experts;
- *Symmetry*: for any permutation in the set of the experts, we have the same aggregate;
- *Invariance with respect to noninformative opinion*: the aggregated assessment of N experts yields the same result as the aggregation of the N opinions with a further noninformative opinion, i.e. with one already implied by the natural extension of the aggregation of the first N;
- *Generalized External Bayesianity*: the aggregation of the original assessments, followed by the coherent extension to a new event, gives the same result as the aggregation of the coherent extensions of the initial assessments.

Moreover, we leave to a future investigation some further basic properties, like for example those proposed by Moral and del Sagrado [23].

5 Weighted aggregation

It is possible to associate different weights to the elements of the joined assessment $(\mathcal{E}, \mathbf{lub})$, as we have already done for precise assessments, reflecting either possible repetitions of the values or different trust on the various sources of information. If we denote by $\mathbf{w} = [w_1, \ldots, w_n]$ such weights, we can adjust the expression of $\Delta(\mathbf{v}, \boldsymbol{\alpha})$ in the optimization problems (11) as

$$\Delta^{\mathbf{w}}(\mathbf{v}, \boldsymbol{\alpha}) := \sum_{i=1}^{n} w_i \alpha(Hi) \left(q_i \ln(\frac{q_i}{v_i}) + (1 - q_i) \ln(\frac{(1 - q_i)}{(1 - v_i)}) \right). \quad (34)$$

We can see directly the effects of this adjustment by a slight modification of Example 2

Example 4 *Let us modify the two original assessments (26) by adding exact overlapping to the previously missing intervals:*

	$C\|A$	$C\|B$	$C\|A \vee B$
lub$'$	[.1, .3]	[.2, .4]	[.6, .8]
lub$''$	[.1, .3]	[.5, .7]	[.6, .8]

. (35)

We can now avoid duplicating identical conditional events with identical intervals in the joint assessment **lub**, *and yet maintain the information of their multiplicity by using the following frequencies weights:*

$\mathcal{E}$	$C\|A$	$C\|B$	$C''\|B''$	$C\|A \vee B$
lub	[.1, .3]	[.2, .4]	[.5, .7]	[.6, .8]
w	2	1	1	2

. (36)

Performing the 8 optimizations with the new objective function (34) under the same constraints (12-15), we obtain as lower-upper envelope of $\mathcal{Q}$

$\mathcal{E}$	$C\|A$	$C\|B$	$C\|A \vee B$
luc	[.1139, .3747]	[.3750, .6242]	[.5355, .6933]

. (37)

Note how the highest weights have "attracted" the aggregation ranges to the corresponding initial assessments.

Weights w_i could be given in an "imprecise" fashion through intervals $[\underline{w_i}, \overline{w_i}]$, especially when they represent trust levels on the sources of information. This does not change the method, but increases the procedure complexity. In fact, in such cases, we can think of the w_i in (34) as further variables in the optimization problems (11-15), with additional constraints $\underline{w_i} \leq w_i \leq \overline{w_i}$. This will affect the numerical expression of the elements inside $\mathcal{Q}$ in (16), but all other considerations will remain the same.

6 Conclusion

The core of our proposal is in the parametric optimization problems (11), based on the discrepancy measure $\Delta(\cdot, \boldsymbol{\alpha})$. Such discrepancy was originally proposed in [4] by a generalization of the logarithmic scoring rule to partial conditional assessments, and has been used to adjust precise evaluations. In [6] we have extended its use to merge different sources of information, and now to correct incoherent imprecise conditional probability assessments.

We have seen through examples (1-4) that the procedure to correct incoherent assessments can be properly extended to aggregate different opinions and generalized to embed importance weights of each assessment. Effectiveness changes if the joint assessment has a coherent least commitment aggregation or not. In fact, if the lower-upper envelope of the union of the opinions turned out to be coherent, our procedure weakens its peculiarity and reduces to the so called "unanimity rule". Anyhow, our proposal is meaningful in the situations when most of the known rules do not apply. In fact, our procedure applies also when the domains of the opinions do not coincide and the numerical parts are strongly inconsistent, so that the aggregation turns out to be a reasonable compromise between the elicited values and the consistency requirement.

This paper reflects just a preliminary study, because, as already mentioned, theoretical aspects will have to be fixed. To begin with, we need to investigate the presumed uniqueness of the assessments induced by the optimal solutions of the parametric optimization problems (11). In fact, the same method applies also when the solution is not unique, but some operational troubles could appear.

Another open problem is about complexity. The check of coherence is already a NP-complete problem "per se". As a consequence, our parametric nonlinear optimization problems (11 - 15) are even harder. Modern optimization tools like GAMS make medium-size problems treatable with some tens of events. One would need heuristic procedures for larger domain problems.

Yet another important further investigation would be to study the relationships with other aggregation rules, in particular comparing properties, and characterizing possible coincidences.

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