Multivariate Models and Confidence Intervals: A Local Random Set Approach

Thomas Fetz
Unit for Engineering Mathematics
University of Innsbruck, Austria
Thomas.Fetz@uibk.ac.at

Abstract

This article is devoted to the propagation of families of confidence intervals obtained by non-parametric methods through multivariate functions comprising the semantics of confidence limits. At fixed confidence level, local random sets are defined whose aggregation admits the calculation of upper probabilities of events. In the multivariate case, a number of ways of combinations is highlighted to encompass independence and unknown interaction using random set independence and Fréchet bounds. For all cases we derive formulas for the corresponding upper probabilities and elaborate how they relate. The methods are exemplified by means of an example from structural mechanics.

Keywords. Confidence intervals, non-parametric models of uncertainty, random sets, fuzzy sets, upper probability, independence, unknown interaction, Fréchet bounds.

1 Introduction

In order to render models of *imprecise probability theory* operative, their *semantics* have to be developed. It has been observed [5, 10, 11] that the idea of *confidence limits* can provide a workable basis for constructing imprecise probability models. In particular, it has been argued in [12, 13] that random sets constructed by Tchebycheff's inequality can serve as a non-parametric model of the variability of a parameter, given its mean value and variance as sole information.

This article develops the concept of using confidence limits for estimating upper and lower probabilities of events. While the papers [5, 12, 13] addressed the univariate case only, it is demonstrated in [10] how to generate joint fuzzy sets from families of marginal confidence intervals using the product t-norm for independence and t-norms based on Fréchet bounds for unknown dependency. In this paper we demonstrate how multivariate input can be treated using a *local random set* approach.

Suppose we are given confidence intervals I_α of some parameter at level α, $0 < \alpha \leq 1$. Then the probability of I_α is bigger than $1 - \alpha$, while the probability of its complement is less than α. The key idea is to define local random sets at level α, formed by I_α and I_α^{c} with weights consistent with the confidence limits. In this way, the upper probability of an event A can be computed as the smallest α for which A lies outside the confidence interval I_α. This procedure gives a conclusive interpretation of upper probabilities in terms of confidence limits.

The plan of this paper is as follows:

In Section 2 families of non-parametric confidence intervals are generated by means of Tchebycheff's inequality.

In Section 3 we introduce the concept of local random sets and its semantics.

In Section 4 it is described how to propagate this kind of uncertainty through univariate functions and it is shown that the local random set approach is consistent with the fuzzy and random set approaches.

In Section 5 we address the multivariate case and generate local joint random sets in various ways consistent with the confidence interpretation. This leads to different estimates for the upper probabilities of events. We derive computational formulas for all cases and show how the results relate to each other and to random set and fuzzy set independence and to the case where nothing is known about how the variables interact.

In Section 6, the method is applied to compute upper distribution functions for the limit state of a beam bedded on two springs where the uncertainty of the spring constants is modelled by families of confidence intervals.

2 Non-parametric models of the variability of a parameter X

In this article we model the variability of a parameter X by a family $\mathbf{I}$ of non-parametric confidence intervals I_α using Tchebycheff's inequality, cf. [5, 13].

Let a random variable X be given with expectation $\mu = \mathsf{E}(X)$ and variance $\sigma^2 = \mathsf{V}(X)$. *Tchebycheff's inequality*

$$P(|X-\mu| > \frac{\sigma}{\sqrt{\alpha}}) \leq \alpha, \quad \alpha \in (0,1]$$

leads to *non-parametric confidence intervals*

$$I_\alpha = \left[\mu - \frac{\sigma}{\sqrt{\alpha}}, \mu + \frac{\sigma}{\sqrt{\alpha}}\right], \quad \alpha \in (0,1]$$

for the variability of X at confidence level $1-\alpha$, given its expectation and variance as sole information. This follows from the fact that the complement I_α^c of I_α is the set used as the argument of P in Tchebycheff's inequality and by

$$P(I_\alpha^\mathsf{c}) \leq \alpha, \quad P(I_\alpha) = 1 - P(I_\alpha^\mathsf{c}) \geq 1-\alpha. \qquad (1)$$

Then the confidence we have in I_α is $1-\alpha$ or greater. All these confidence intervals together are a family denoted by $\mathbf{I} = \{I_\alpha\}_{\alpha\in(0,1]}$ and they are nested, since $I_\alpha \supseteq I_\beta$ if $\alpha \leq \beta$. This property will be also important in the multivariate case later on. A family $\mathbf{I}$ is visualized by plotting in Fig. 1 the interval bounds of I_α, $\alpha \in (0,1]$, at levels α.

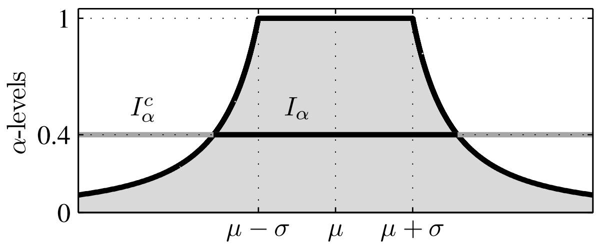

Figure 1: Example of a family $\mathbf{I}$.

3 The univariate case

Let a family $\mathbf{I}$ of confidence intervals I_α, $\alpha \in (0,1]$, generated as in the previous Section be given.

3.1 Local random sets at level α

We assume that $\alpha \in (0,1]$ is fixed. Equipping the two intervals I_α and I_α^c with weights $m(I_\alpha)$ and $m(I_\alpha^\mathsf{c})$ we get a finite random set. The possible values of these weights are determined by

$$m(I_\alpha) = P(I_\alpha) \text{ and } m(I_\alpha^\mathsf{c}) = P(I_\alpha^\mathsf{c})$$

and the inequalities (1) where the weight $m(I_\alpha)$ of I_α corresponds to the confidence we have in the set I_α. We call such a random set corresponding to a certain level α *local random set.*

For an arbitrary event A there are three possibilities for the relations to the two focal sets. These relations and the corresponding upper probabilities $\overline{P}_\alpha$ are shown in the following table:

	Cases	$\overline{P}_\alpha(A) \in$
(i)	$A \cap I_\alpha = \varnothing$	$[0,\alpha]$
(ii)	$A \cap I_\alpha^\mathsf{c} = \varnothing$	$[1-\alpha, 1]$
(iii)	$A \cap I_\alpha \neq \varnothing$ and $A \cap I_\alpha^\mathsf{c} \neq \varnothing$	1

The *local upper probability* $\overline{P}_\alpha(A)$ at level α for an event A is obtained by

$$\overline{P}_\alpha(A) = m(I_\alpha)\,\chi(A\cap I_\alpha \neq \varnothing) + \\ + m(I_\alpha^\mathsf{c})\,\chi(A\cap I_\alpha^\mathsf{c} \neq \varnothing)$$

where $\chi : \mathbb{R} \to \{0,1\}$ is the indicator function. Here the upper probabilities are intervals because of the inequalities (1) for the weights.

If A has the role of the "bad" and undesired event, case (i) is the most interesting one, because its meaning is:

If A is outside the confidence interval I_α at confidence level $1-\alpha$, then we can say for sure that A occurs only with probability α, at most.

To avoid interval-valued weights and upper probabilities we take always the upper bounds of $\overline{P}_\alpha$ in the above table, that means

$$\overline{P}_\alpha(A) := \begin{cases} \alpha & \text{if } A \cap I_\alpha = \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

Then we are on the safe side in all three cases.

In general we are not in the interesting case (i) for a given A, but we can try to achieve the situation of case (i) by increasing α. On the other hand, if we are already in case (i), we should try to decrease α to get a smaller upper probability $\overline{P}_\alpha$. This leads to the following rule for the upper probability $\overline{P}(A)$, cf. Fig. 2:

Find the confidence interval I_{α^*} with the smallest $\alpha^* \in (0,1]$ among those confidence intervals I_α with $I_\alpha \cap A = \varnothing$. Then $\overline{P}(A) = \alpha^*$. If we do not find such an interval I_{a^*}, then $\overline{P}(A) = 1$.

With $\inf\{\varnothing\} = 1$ to encompass the case where no I_{a^*} can be found, we get the following formula for the upper probability:

$$\overline{P}(A) = \inf\{\alpha \in (0,1] : I_\alpha \cap A = \varnothing\} = \alpha^*. \tag{2}$$

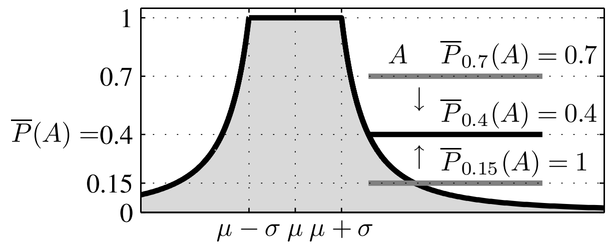

Figure 2: Computation of $\overline{P}(A)$.

3.2 Interpretation of I as a random set and fuzzy set

Together with the uniform distribution on the interval $(0,1]$, the family $\mathbf{I} = \{I_\alpha\}_{\alpha\in(0,1]}$ of confidence intervals is an infinite random set [3, 4, 11]. Note that now all $I_\alpha \in \mathbf{I}$ together play the role of focal sets and not only two sets I_α, I_α^c for fixed α as before. Then for the upper probability $\overline{P}(A)$ (or Plausibility) we get

$$\overline{P}(A) = \mathrm{Pl}(A) = \int_{\beta:\ I_\beta\cap A\neq\varnothing} \mathrm{d}\beta = 1 - \int_{\beta:\ I_\beta\cap A=\varnothing} \mathrm{d}\beta = $$
$$= 1 - \int_{\inf\{\beta:\ I_\beta\cap A=\varnothing\}=\alpha^*}^{1} \mathrm{d}\beta = \alpha^*,$$

because the confidence intervals I_β are nested.

Now we interpret the family $\mathbf{I}$ of nested confidence intervals I_α as fuzzy numbers [14] defined by the α-level sets I_α. The membership function μ is given by the endpoints of the intervals I_α as in Fig. 1. Then the upper probability $\overline{P}(A)$ (or Possibility) is given by

$$\begin{aligned}\overline{P}(A) &= \mathrm{Pos}(A) = \sup\{\mu(x) :\ x \in A\} = \\ &= \sup\{\alpha \in (0,1] :\ I_\alpha \cap A \neq \varnothing\} = \\ &= \inf\{\alpha \in (0,1] :\ I_\alpha \cap A = \varnothing\} = \alpha^*\end{aligned}$$

where $\sup\{\varnothing\} = 0$.

So all three interpretations lead to the same result for the upper probability $\overline{P}(A)$.

4 Propagation of uncertainty trough a univariate function g

4.1 Preliminaries

Let a continuous function

$$g : D \subseteq \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto g(x)$$

and a family $\mathbf{I}$ of confidence intervals I_α be given where we assume that $I_\alpha \subseteq D$ which we achieve simply by truncating I_α if necessary.

Further we are using in the following that

$$\begin{aligned}&P(g(X) \in A) = P(X \in g^{-1}(A)),\\ &I_\alpha \cap g^{-1}(A) = \varnothing \Longleftrightarrow g(I_\alpha) \cap A = \varnothing \text{ and}\\ &I_\alpha \cap g^{-1}(A) \neq \varnothing \Longleftrightarrow g(I_\alpha) \cap A \neq \varnothing\end{aligned}$$

where $g(I_\alpha) = \{g(x) :\ x \in I_\alpha\}$ is the image of I_α under g and $g^{-1}(A) = \{x :\ g(x) \in A\}$ the inverse image of A.

Now we compute $\overline{P}(g(X) \in A)$ for the local random set approach and show that we get the same result as for the random set and for the fuzzy set interpretation.

4.2 Local random set approach

For the local random set approach we have

$$\begin{aligned}\overline{P}(g(X) \in A) &= \overline{P}(X \in g^{-1}(A)) = \\ &= \inf\{\alpha \in (0,1] :\ I_\alpha \cap g^{-1}(A) = \varnothing\} = \\ &= \inf\{\alpha \in (0,1] :\ g(I_\alpha) \cap A = \varnothing\} = \alpha^*.\end{aligned}$$

The only difference to Eq. (2) is that now $g(I_\alpha)$ is used instead of I_α. This motivates the definition

$$g(\mathbf{I}) = \{g(I_\alpha)\}_{\alpha\in(0,1]}$$

which is the family of the images of all confidence intervals. Propagating $\mathbf{I}$ through a function in the univariate case means simply replacing $\mathbf{I}$ by $g(\mathbf{I})$ and applying formula (2), cf. Fig. 3.

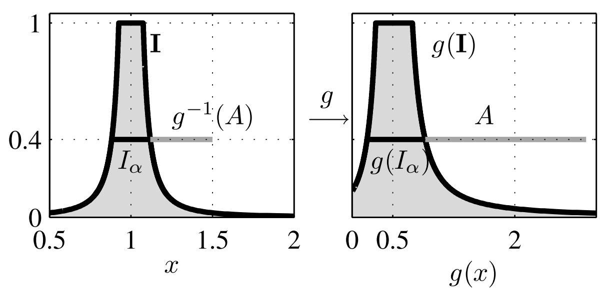

Figure 3: Computation of $\overline{P}(g(X) \in A)$.

4.3 Random set and fuzzy set approach

By the arguments presented in the preliminaries and in Section 3 we get for the random set interpretation

$$\overline{P}(g(X) \in A) = \mathrm{Pl}(g(X) \in A) = \int_{\beta:\ g(I_\beta)\cap A\neq\varnothing} \mathrm{d}\beta =$$
$$= 1 - \int_{\beta:\ g(I_\beta)\cap A=\varnothing} \mathrm{d}\beta = 1 - \int_{\alpha^*}^{1} \mathrm{d}\beta = \alpha^*$$

since again the $g(I_\beta)$ are nested. Further we get for the fuzzy set interpretation of $\mathbf{I}$ with α-level sets I_α:

$$\begin{aligned}\overline{P}(g(X) \in A) &= \mathrm{Pos}(g(X) \in A) = \\ &= \sup\{\alpha \in (0,1] : \; g(I_\alpha) \cap A \neq \varnothing\} \\ &= \inf\{\alpha \in (0,1] : \; g(I_\alpha) \cap A = \varnothing\} = \alpha^*.\end{aligned}$$

4.4 Summary

As we have seen the local random set approach preserves the method of searching for the "best" confidence interval when applying a univariate function g.

More important is, that the result is consistent with the random set and fuzzy set interpretation of the family of confidence intervals. But this is only true in the univariate case. It is a wellknown fact that the random set and the fuzzy set approach lead to different results in the multivariate case which will also have consequences for the local random set version.

5 The multivariate case

Here we assume that for n random variables $X_1, \ldots, X_n$ families $\mathbf{I}_1, \ldots, \mathbf{I}_n$ of confidence intervals are given. Then we have to determine the joint uncertainty of all these variables which will be done by means of local joint random sets obtained by combining confidence intervals $I_{1,\alpha_1} \in \mathbf{I}_1, \ldots, I_{n,\alpha_n} \in \mathbf{I}_n$.

The goal of this and the next Section is to get a formula similar to the univariate version

$$\overline{P}(g(X) \in A) = \inf\{\alpha \in (0,1] : \; g(I_\alpha) \cap A = \varnothing\}.$$

But such a formula will not be uniquely defined because we have several possibilities of choice

- for the set of confidence intervals considered to be combined and
- for the weights used for the local joint random set.

5.1 Combination of marginal confidence intervals

Let the *joint confidence set* $J_{\boldsymbol{\alpha}}$ be given by

$$J_{\boldsymbol{\alpha}} = I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$. Then $\mathbf{J} = \{J_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in S}$ is the family of all joint confidence sets depending on which set S of indices $\boldsymbol{\alpha}$ is considered.

If $S = S_{\mathrm{R}} = (0,1]^n$ then all possible combinations of confidence intervals are used, exactly as the joint focal sets are generated for random set independence.

A second possibility is to combine only confidence intervals of the same level α similar to the combination of the α-level sets for fuzzy set independence. In this case we have the set

$$S = S_{\mathrm{F}} = \{\boldsymbol{\alpha} \in (0,1]^n : \; \alpha_1 = \alpha_2 = \cdots = \alpha_n\} \subseteq S_{\mathrm{R}}$$

which has the advantage that the number of joint confidence sets does not grow with the number of variables. For simplification we will then also use the notation

$$\mathbf{J} = \{J_\alpha\}_{\alpha \in [0,1]} = \{I_{1,\alpha} \times \cdots \times I_{n,\alpha}\}_{\alpha \in [0,1]}$$

for the family of joint confidence sets.

5.2 Local joint random sets

For two variables X_1 and X_2 the combination of a confidence interval $I_{1,\alpha_1} \in \mathbf{I}_1$ at confidence level $1-\alpha_1$ with a confidence interval $I_{2,\alpha_2} \in \mathbf{I}_2$ at confidence level $1-\alpha_2$ means to generate a local joint random set with focal sets

$$I_{1,\alpha_1} \times I_{2,\alpha_2}, \; I^{\mathsf{c}}_{1,\alpha_1} \times I_{2,\alpha_2}, \; I_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2}, \; I^{\mathsf{c}}_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2}$$

from the marginal local random set at level α_1 with focal sets I_{1,α_1}, $I^{\mathsf{c}}_{1,\alpha_1}$ for the first variable and from the marginal local random set at level α_2 with focal sets I_{2,α_2}, $I^{\mathsf{c}}_{2,\alpha_2}$ for the second one, cf. Fig. 4. The focal set $J_{\boldsymbol{\alpha}} = I_{1,\alpha_1} \times I_{2,\alpha_2}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, is then the joint confidence set.

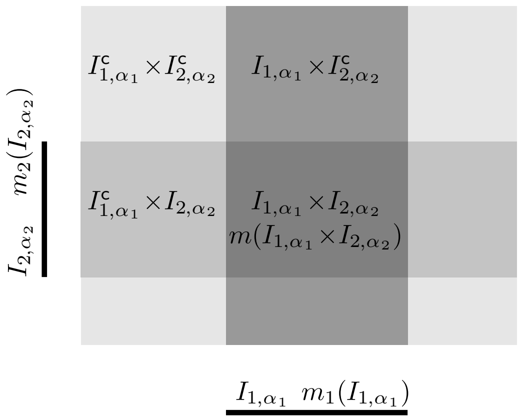

Figure 4: Joint focal sets $I_{1,\alpha_1} \times I_{2,\alpha_2}$, $I^{\mathsf{c}}_{1,\alpha_1} \times I_{2,\alpha_2}$, $I_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2}$ and $I^{\mathsf{c}}_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2}$.

Computation of the local upper probability $\overline{P}_{\boldsymbol{\alpha}}(A)$:

In the following we do not care about how an event A with empty intersection with the joint confidence set $I_{1,\alpha_1} \times I_{2,\alpha_2}$ hits the remaining three focals sets. We assume the worst case (hitting all three focals sets), that means

$$\begin{aligned}\overline{P}_{\boldsymbol{\alpha}}(A) &= m(I_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2}) + m(I_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2}) + \\ &\qquad + m(I^{\mathsf{c}}_{1,\alpha_1} \times I_{2,\alpha_2}) \\ &= 1 - m(I_{1,\alpha_1} \times I_{2,\alpha_2}) = \overline{P}((I_{1,\alpha_1} \times I_{2,\alpha_2})^{\mathsf{c}})\end{aligned}$$

which relieves us from computing images of sets where the complements are involved.

For n variables we have then

$$\overline{P}_{\boldsymbol{\alpha}}(A) = 1 - m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})$$

for $(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) \cap A = \varnothing$.

5.3 The local joint weight

The main task is to determine the weight $m(I_{1,\alpha_1} \times I_{2,\alpha_2})$ of the joint confidence set $I_{1,\alpha_1} \times I_{2,\alpha_2}$ which represents the confidence we have in this set.

This weight is not uniquely determined, because joint probability distributions are not unique in general. The weights of all four joint focal sets (see Fig. 4) has to be chosen in a way that the horizontal and vertical sums in the following table lead to the marginal weights m_i which are either α_1 or $1-\alpha_1$ for the first variable and either α_2 or $1-\alpha_2$ for the second one:

$m_2(I^{\mathsf{c}}_{2,\alpha_2}) = \alpha_2$	$m(I_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2})$	$m(I^{\mathsf{c}}_{1,\alpha_1} \times I^{\mathsf{c}}_{2,\alpha_2})$
$m_2(I_{2,\alpha_2}) = 1-\alpha_2$	$m(I_{1,\alpha_1} \times I_{2,\alpha_2})$	$m(I^{\mathsf{c}}_{1,\alpha_1} \times I_{2,\alpha_2})$
	$m_1(I_{1,\alpha_1}) = 1-\alpha_1$	$m_1(I^{\mathsf{c}}_{1,\alpha_1}) = \alpha_1$

5.3.1 Random set independence

In the case of random set independence the weight of the joint confidence set is given by the product of the marginal weights. For n variables we have then

$$m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) = \prod_{i=1}^{n} m_i(I_{i,\alpha_i}) = \prod_{i=1}^{n} (1-\alpha_i)$$

which leads to the local upper probability

$$\overline{P}_{\boldsymbol{\alpha}}(A) = 1 - m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) = 1 - \prod_{i=1}^{n}(1-\alpha_i)$$

if $(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n}) \cap A = \varnothing$.

If it is known that the uncertain variables are independent, random set independence is one possibility to take the independence of the variables into account. We note that there are other notions of independence such as strong independence and epistemic independence [2, 6, 7, 8].

5.3.2 Lower and upper bounds for the focal weights $m(I_{1,\alpha_1} \times \cdots \times I_{n,\alpha_n})$

Using the bounds of Fréchet [9] for joint probability distributions we get in the 2-dimensional case for the joint weight $m(I_{1,\alpha_1} \times I_{2,\alpha_2})$

$$\begin{aligned}\max(m(I_{1,\alpha_1}) + m(I_{2,\alpha_2}) - 1, 0) &\le m(I_{1,\alpha_1} \times I_{2,\alpha_2}) \le \\ &\le \min(m(I_{1,\alpha_1}), m(I_{2,\alpha_2}))\end{aligned}$$

and with $m(I_{i,\alpha_i}) = 1 - \alpha_i$

$$\begin{aligned}\max(1-\alpha_1-\alpha_2, 0) &\le m(I_{1,\alpha_1} \times I_{2,\alpha_2}) \le \\ &\le \min(1-\alpha_1, 1-\alpha_2).\end{aligned}$$

Further using that the local upper probability $\overline{P}_{\boldsymbol{\alpha}}(A) = 1 - m(I_{1,\alpha_1} \times I_{2,\alpha_2})$ for $(I_{1,\alpha_1} \times I_{2,\alpha_2}) \cap A = \varnothing$ leads to lower and upper bounds

$$\max(\alpha_1, \alpha_2) \le \overline{P}_{\boldsymbol{\alpha}}(A) \le \min(\alpha_1 + \alpha_2, 1)$$

for $\overline{P}_{\boldsymbol{\alpha}}(A)$.

With Frechet's version of the inequality for n variables we get then the bounds

$$\max_{i=1,\ldots,n}(\alpha_i) \le \overline{P}_{\boldsymbol{\alpha}}(A) \le \min(\alpha_1 + \cdots + \alpha_n, 1).$$

We use these bounds if nothing is known about how the uncertain variables interact.

5.4 Levels of the joint confidence set

These different approaches have only an influence on the level of the joint confidence sets, but not on the sets itself.

For the three different approaches (random set independence, lower bound and upper bound) we have different levels described by the level function

$$\ell(\boldsymbol{\alpha}) = \begin{cases} \max\limits_{i=1,\ldots,n}(\alpha_i) & \text{lower bound,} \\ 1 - \prod_{i=1}^{n}(1-\alpha_i) & \text{random set independence,} \\ \min(\alpha_1 + \cdots + \alpha_n, 1) & \text{upper bound} \end{cases}$$

which leads to the upper probability

$$\overline{P}^{S}_{\ell}(A) = \inf_{\boldsymbol{\alpha} \in S} \{\ell(\boldsymbol{\alpha}) : J_\alpha \cap A = \varnothing\}$$

where the subscript ℓ indicates the level function and the superscript S the set of the $(\alpha_1, \ldots, \alpha_n)$ considered.

5.5 Propagating uncertainty through a multivariate function g

Let a continuous multivariate function

$$g : D \subseteq \mathbb{R}^n \to \mathbb{R} : x \mapsto g(x)$$

be given.

Using the same ideas as in the univariate case we get now the desired formula for the upper probability

$$\begin{aligned}\overline{P}_\ell^S&(g(X)\in A)=\\&=\inf_{\boldsymbol{\alpha}\in S}\{\ell(\boldsymbol{\alpha}):\ J_{\boldsymbol{\alpha}}\cap g^{-1}(A)=\varnothing\}=\\&=\inf_{\boldsymbol{\alpha}\in S}\{\ell(\boldsymbol{\alpha}):\ g(J_{\boldsymbol{\alpha}})\cap A=\varnothing\}.\end{aligned}$$

We note that this is the same formula as in the univariate case with the only difference that the level $\ell(\alpha)$ of the resulting interval $g(J_{\boldsymbol{\alpha}})=g(I_{1,\alpha_1}\times\cdots\times I_{n,\alpha_n})$ may change according to the chosen level function ℓ and that the upper probability depends on the set of confidence intervals considered for combination which is indicated again by ℓ and S.

5.6 Notations

We introduce the following notations for the upper probability $\overline{P}_\ell^S(A)=\inf_{\boldsymbol{\alpha}\in S}\{\ell(\boldsymbol{\alpha}):\ J_\alpha\cap A=\varnothing\}$ depending on ℓ and S.

If all possible combinations of confidence intervals are allowed, $S=S_{\mathrm{R}}$, we indicate this by the superscript R:

Notation	level $\ell(\boldsymbol{\alpha})$	
$\overline{P}_{\mathrm{lower}}^{\mathrm{R}}$	$\max\limits_{i=1,\ldots,n}(\alpha_i)$	lower Fréchet bound
$\overline{P}_{\mathrm{indep}}^{\mathrm{R}}$	$1-\prod_{i=1}^n(1-\alpha_i)$	random set independence
$\overline{P}_{\mathrm{upper}}^{\mathrm{R}}$	$\min\left(\sum_{i=1}^n\alpha_i,1\right)$	upper Fréchet bound

If we consider only combinations of confidence intervals of the same level α, $S=S_{\mathrm{F}}$, we indicate this by the superscript F:

Notation	level $\ell(\alpha)$	
$\overline{P}_{\mathrm{lower}}^{\mathrm{F}}$	α	lower Fréchet bound
$\overline{P}_{\mathrm{indep}}^{\mathrm{F}}$	$1-(1-\alpha)^n$	random set independence
$\overline{P}_{\mathrm{upper}}^{\mathrm{F}}$	$\min(n\alpha,1)$	upper Fréchet bound

Now we recall the definitions of the upper probabilities for random set independence, fuzzy set independence and unknown interaction in the multivariate case where the notations are given in the following table:

Notation	
$\overline{P}_{\mathrm{R}}$	random set independence
$\overline{P}_{\mathrm{F}}$	fuzzy set independence
$\overline{P}_{\mathrm{U}}$	unknown interaction

The upper probability for random set independence (joint plausibility measure) is defined by

$$\overline{P}_{\mathrm{R}}(A)=\int\limits_{(0,1]^n}\chi(J_{\boldsymbol{\beta}}\cap A\neq\varnothing)\,\mathrm{d}\boldsymbol{\beta}$$

where the $J_{\boldsymbol{\beta}}=I_{1,\beta_1}\times\cdots\times I_{n,\beta_n}$ has the meaning of joint focal sets.

The upper probability for fuzzy set independence (joint possibility measure) is given by

$$\overline{P}_{\mathrm{F}}(A)=\sup\{\alpha\in(0,1]:\ J_\alpha\cap A\neq\varnothing\}$$

where $J_\alpha=I_{1,\alpha}\times\cdots\times I_{n,\alpha}$ are now the joint α-level sets.

In the case where we do not know how the variables are correlated or interact the upper probability for unknown interaction is defined by

$$\overline{P}_{\mathrm{U}}(A)=\sup\{P(A):\ P\in\mathcal{M}_{\mathrm{U}}\}$$

where $\mathcal{M}_{\mathrm{U}}$ is the biggest set of all joint probability measures generated by marginal probability measures compatible with the families of confidence intervals.

For upper distribution functions defined by

$$\overline{F}_\ell^S(x)=\overline{P}_\ell^S((\infty,x])$$

we use the analogous notation as presented in the above tables, e.g. $\overline{F}_{\mathrm{indep}}^R$ is the upper distribution function for $\ell(\boldsymbol{\alpha})=1-\prod_{i=1}^n(1-\alpha_i)$ and $S=S_{\mathrm{R}}$.

5.7 The ordering of the upper probabilities

With $\boldsymbol{\beta}\leq\boldsymbol{\alpha}$ defined by $\beta_i\leq\alpha_i$, $i=1,\ldots,n$, we have the order relation

$$J_{\boldsymbol{\alpha}}\subseteq J_{\boldsymbol{\beta}}\Longleftrightarrow\boldsymbol{\beta}\leq\boldsymbol{\alpha}$$

since all $\mathbf{I}_i$ are families of nested confidence intervals.

Let an event A be given. Then we have always an $\boldsymbol{\alpha}\in S_{\mathrm{R}}$ such that

$$\overline{P}_\ell^{S_{\mathrm{R}}}(A)=\inf_{\boldsymbol{\beta}\in S_{\mathrm{R}}}\{\ell(\boldsymbol{\beta}):\ J_{\boldsymbol{\beta}}\cap A=\varnothing\}=\ell(\boldsymbol{\alpha}),$$

because all level functions ℓ are continuous.

Inspired by a figure in [1] used in a different context we define for above A and $\boldsymbol{\alpha}$ the sets :

$$\begin{aligned}S_{\mathrm{hit}}(A)&=\{\boldsymbol{\alpha}\in S_{\mathrm{R}}:\ J_{\boldsymbol{\alpha}}\cap A\neq\varnothing\},\\\underline{S}(\boldsymbol{\alpha})&=\{\boldsymbol{\beta}\in S_{\mathrm{R}}:\ \ell(\boldsymbol{\beta})\leq\ell(\boldsymbol{\alpha})=\overline{P}_\ell^{S_{\mathrm{R}}}(A)\}\end{aligned}$$

and

$$\overline{S}(\boldsymbol{\alpha})=(0,1]^n\setminus((\alpha_1,1]\times\cdots\times(\alpha_n,1]),$$

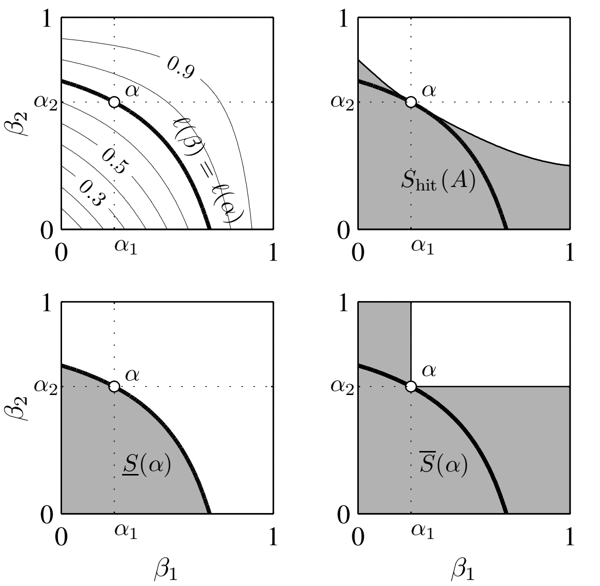

Figure 5: Contourlines of $1-(1-\alpha_1)(1-\alpha_2)$ and the sets $S_{\rm hit}(A)$, $\underline{S}(\boldsymbol{\alpha})$ and $\overline{S}(\boldsymbol{\alpha})$ for a given $\boldsymbol{\alpha}$.

cf. Fig. 5.

For $\boldsymbol{\alpha} \in S_{\rm hit}(A)$ the set $S_{\rm hit}(A)$ has the property

$$\boldsymbol{\beta} \leq \boldsymbol{\alpha} \implies \boldsymbol{\beta} \in S_{\rm hit}(A).$$

Since all level functions are increasing in all directions $\underline{S}(\boldsymbol{\alpha})$ and obviously $\overline{S}(\boldsymbol{\alpha})$ also have this property.

Then we have $\underline{S}(\boldsymbol{\alpha}) \subseteq S_{\rm hit}(A) \subseteq \overline{S}(\boldsymbol{\alpha})$. See Fig. 5.

Since $S_{\rm F} \subseteq S_{\rm R}$ we have always $\overline{P}_\ell^{S_{\rm R}}(A) \leq \overline{P}_\ell^{S_{\rm F}}$.

5.7.1 $\overline{P}_{\rm R}(A) \leq \overline{P}^{\rm R}_{\rm indep}(A) \leq \overline{P}^{\rm F}_{\rm indep}(A)$

Let $\boldsymbol{\alpha} \in S_{\rm R}$, such that

$$\overline{P}^{\rm R}_{\rm indep}(A) = \ell(\boldsymbol{\alpha}) = 1 - \prod_{i=1}^{n}(1-\alpha_i).$$

Then

$$\begin{aligned}\overline{P}_{\rm R}(A) &= \int_{S_{\rm hit}(A)} \mathrm{d}\boldsymbol{\beta} \leq \int_{\overline{S}(\boldsymbol{\alpha})} \mathrm{d}\boldsymbol{\beta} = \\ &= 1 - \prod_{i=1}^{n}(1-\alpha_i) = \overline{P}^{\rm R}_{\rm indep}(A).\end{aligned}$$

5.7.2 $\overline{P}_{\rm U}(A) \leq \overline{P}^{\rm R}_{\rm upper}(A) \leq \overline{P}^{\rm F}_{\rm upper}(A)$

Let $p_{[0,1]}$ the probability measure representing the uniform distribution on $[0,1]$, $\mathcal{M}'_{\rm U}$ the set of all probability measures p on $(0,1]^n$ whose marginals are $p_{[0,1]}$ and $\overline{p}(S) = \sup\{p(S) :\ p \in \mathcal{M}'_{\rm U}\}$. Then we have

$$\overline{p}(S_{\rm hit}(A)) = \overline{P}_{\rm U}(A) \leq \overline{p}(\overline{S}(\boldsymbol{\alpha})).$$

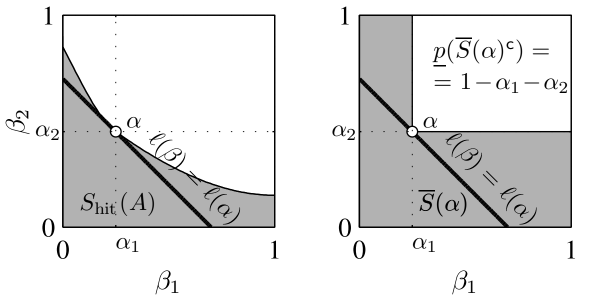

Figure 6: 2-dimensional visualization of the proof in Sec. 5.7.2

The least probability we can concentrate in $\overline{S}(\boldsymbol{\alpha})^{\mathsf{c}}$ is given by

$$\begin{aligned}\underline{p}(\overline{S}(\boldsymbol{\alpha})^{\mathsf{c}}) &= \max\left(\sum_{i=1}^{n} p_{[0,1]}((\alpha_i,1]) - (n-1), 0\right) = \\ &= \max\left(\sum_{i=1}^{n}(1-\alpha_i) - (n-1), 0\right)\end{aligned}$$

using the lower Fréchet bound which leads to

$$\begin{aligned}\overline{p}(\overline{S}(\boldsymbol{\alpha})) &= 1 - \underline{p}(\overline{S}(\boldsymbol{\alpha})^{\mathsf{c}}) = \\ &= \min\left(\sum_{i=1}^{n}\alpha_i, 1\right) = \overline{P}^{\rm R}_{\rm upper}(A),\end{aligned}$$

cf. Fig. 6 for the 2-dimensional case.

5.7.3 $\overline{P}^{\rm R}_{\rm lower}(A) = \overline{P}^{\rm F}_{\rm lower}(A) = \overline{P}_{\rm F}(A)$

First we show that $\overline{P}^{\rm F}_{\rm lower}(A) = \overline{P}_{\rm F}(A)$:

$$\begin{aligned}\overline{P}_{\rm F}(A) &= \sup_{\alpha\in(0,1]} \{J_\alpha \cap A \neq \varnothing\} = \\ &= \inf_{\alpha\in(0,1]=S_{\rm F}} \{J_\alpha \cap A = \varnothing\} = \overline{P}^{\rm F}_{\rm lower}(A)\end{aligned}$$

with $J_\alpha = I_{1,\alpha} \times \cdots \times I_{n,\alpha}$ which is both the joint confidence at level α and the corresponding joint α-level set.

Again let $\boldsymbol{\alpha} \in S_{\rm R}$, such that

$$\overline{P}^{\rm R}_{\rm lower}(A) = \ell(\boldsymbol{\alpha}) = \max_{i=1,\dots,n}(\alpha_i) =: \alpha.$$

But then also $(\alpha,\dots,\alpha) \in \underline{S}(\boldsymbol{\alpha})$ and $\overline{P}^{\rm F}_{\rm lower}(A) = \alpha$ which proves the first equality.

5.8 The special case $S = S_{\rm F}$

In the case of $S = S_{\rm F}$ the joint confidence sets are nested. Let

$$G_\alpha = g(J_\alpha),\ \alpha \in (0,1]$$

be the image of the joint confidence set J_α under g. The index α does correspond only to the case where $\ell(\alpha) = \alpha$. But if we lift the images of the joint confidence sets to the right level by the transformation $H_\alpha = G_{\ell^{-1}(\alpha)}$, $\alpha = 1, \ldots, n$, we get the family

$$\mathbf{H}_\ell = \{H_\alpha\}_{\alpha \in (0,1]}$$

where ℓ indicates the level function used for the transformation. Then the upper probability corresponding to ℓ is simply obtained by

$$\overline{P}_\ell(A) = \inf_{\alpha \in (0,1]} \{H_\alpha \cap A = \varnothing\}$$

as in Section 3 and 4 where no ℓ appears in the formula. In Fig. 7 families $\mathbf{H}_\ell$ are plotted for the three different level functions ℓ presented in this paper.

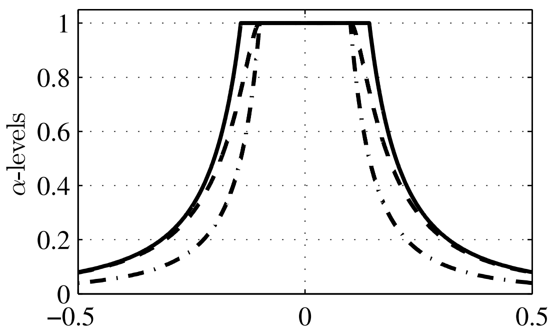

Figure 7: An example of families $\mathbf{H}_\ell$ for ℓ corresponding to the upper bound (solid) and lower bound (dash-dotted) and for random set independence (dashed).

6 A Numerical Example

As a numerical example we consider a beam of length 3 m supported on both ends and additionally bedded on two springs, cf. Fig. 8. The values of the beam rigidity $EI = 1$ kNm^2 and of the equally distributed load $f(x) = 100$ kN/m are assumed to be deterministic, but the values of the two spring constants λ_1 and λ_2 are uncertain.

In this example we assume that the expectations and variances of the two variables λ_1 and λ_2 are given as in the following table.

variable	expectation	variance
λ_1	30	2
λ_2	35	1.5

The corresponding families of confidence intervals generated by means of Tchebycheff's inequality are truncated by the interval $[0, 50]$ and depicted in Fig. 9.

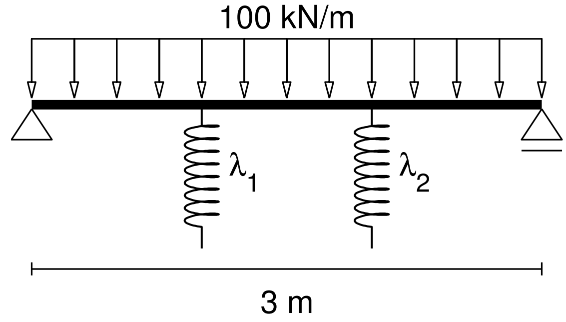

Figure 8: A beam bedded on two springs.

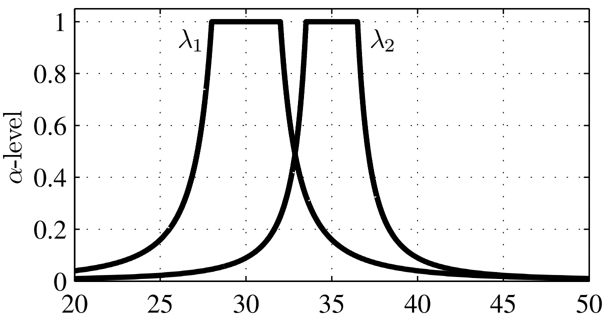

Figure 9: Families of confidence intervals for the two spring constants.

Now we want to compute the upper probability of failure of the beam. The criterion of failure is

$$\max_{x \in [0,3]} |M(x, \lambda_1, \lambda_2)| \geq M_{\mathsf{yield}}$$

where $M(x)$ is the bending moment at point $x \in [0, 3]$ depending on the two spring constants λ_1, λ_2 and $M_{\mathsf{yield}} = 12$ kNm the elastic limit moment. We reformulate the failure criterion as failure function

$$g(\lambda_1, \lambda_2) = M_{\mathsf{yield}} - \max_{x \in [0,3]} |M(x, \lambda_1, \lambda_2)|$$

where now $g(\lambda_1, \lambda_2) \leq 0$ means failure. In Fig. 10 the failure function g is depicted as a contour plot for values $(\lambda_1, \lambda_2) \in [10, 45] \times [10, 45]$ where we can see that g is a concave function in both directions.

Since we want to know if $g(\lambda_1, \lambda_2)$ becomes zero it is sufficient to have only the lower bounds of the images

$$G_{\boldsymbol{\alpha}} = [\underline{G}_{\boldsymbol{\alpha}}, \overline{G}_{\boldsymbol{\alpha}}] = g(J_{\boldsymbol{\alpha}})$$

of the joint confidence sets $J_{\boldsymbol{\alpha}} = \lambda_{1,\alpha_1} \times \lambda_{2,\alpha_2}$. These lower bounds can be easily obtained by minimizing the function values at the vertices of the joint confidence set which is not true for the upper bounds.

The function values $g(\lambda_1, \lambda_2)$ are computed by the finite element method. To omit a large number of function evaluations for a large number of joint confidence

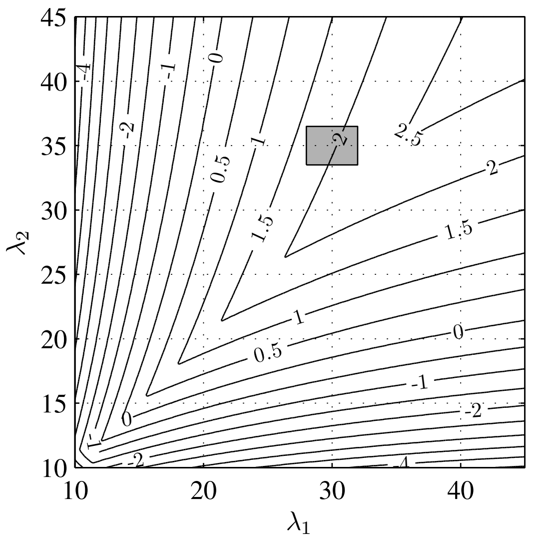

Figure 10: Contour plot of the failure function g. The gray rectangle is the joint confidence set $\lambda_{1,\alpha_1} \times \lambda_{2,\alpha_2}$ for $(\alpha_1, \alpha_2) = (1, 1)$.

sets to be considered we evaluate g on grid points on $[0, 50] \times [0, 50]$ and get $g(\lambda_1, \lambda_2)$ using interpolation.

We get the upper probability distribution functions $\overline{F}^S_\ell$ by

$$\overline{F}^S_\ell(x) = \overline{P}^S_\ell((\infty, x]) = \inf_{\boldsymbol{\alpha} \in S} \{\ell(\boldsymbol{\alpha}) : \underline{G}_{\boldsymbol{\alpha}} > x\}.$$

The results are plotted for $x \in [-0.5, 1.75]$ in Fig. 11. The upper probabilities $\overline{P}^S_\ell((\infty, 0])$ of failure are given by the upper distribution functions at zero.

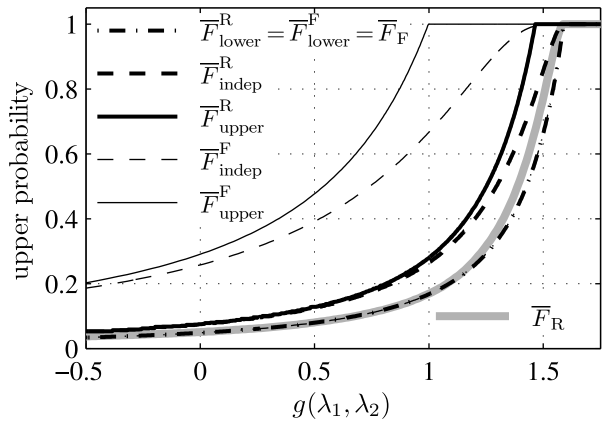

Figure 11: Upper probability distribution functions.

Conclusion

The notion of local random sets was introduced in this article in order to provide a conclusive semantic connection between confidence intervals and random sets. We showed how upper probabilities of events can be calculated from families confidence intervals. The upper probabilities are unique in the univariate case, while in the multivariate case different methods of combinations leading to different upper probabilities are admissible. Further we gave computational formulas for all cases and showed how the resulting upper probabilities are ordered. We demonstrated how the method can be applied in an example from structural mechanics.

References

[1] C. Baudrit and D. Dubois. Comparing methods for joint objective and subjective uncertatinty propagation with an example in a risk assessment. In *F. Gagliardi Cozman, R. Nau, T. Seidenfeld (Eds.): ISIPTA '05, Proceedings of the Fourth International Symposium on Imprecise Probabilities and Their Applications*, Carnegie Mellon University, Pittsburgh, 2005.

[2] I. Couso, S. Moral, and P. Walley. Examples of independence for imprecise probabilities. In G. de Cooman, G. Cozman, S. Moral, and P. Walley, editors, *Proceedings of the first international symposium on imprecise probabilities and their applications*, pages 121–130, Ghent, 1999. Universiteit Gent.

[3] A.P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Stat.*, 38:325–339, 1967.

[4] A.P. Dempster. Upper and lower probabilities generated by a random closed interval. *Ann. Math. Statistics*, 39:957–966, 1968.

[5] D. Dubois, L. Foulloy, G. Mauris, and H. Prade. Probability-possibility transformations, triangular fuzzy sets, and probabilistic inequalities. *Reliable Computing*, 10:273–297, 2004.

[6] Th. Fetz. Sets of joint probability measures generated by weighted marginal focal sets. In *G. de Cooman, T. Fine, T. Seidenfeld (Eds.), ISIPTA'01, Proceedings of the Second Symposium on Imprecise Probabilities and Their Applications*, pages 171–178, Maastricht, 2001. Shaker Publ. BV.

[7] Th. Fetz. Multi-parameter models: rules and computational methods for combining uncertainties. In W. Fellin, H. Lessman, R. Vieider, and M. Oberguggenberger, editors, *Analyzing Uncertainty in Civil Engineering*. Springer, Berlin, 2004.

[8] Th. Fetz and M. Oberguggenberger. Propagation of uncertainty through multivariate functions in the framework of sets of probability measures. *Reliability Engineering and System Safety*, 85(1-3):73–87, 2004.

[9] M. Fréchet. Généralisations du théorème des probabilités totales. *Fundamenta Mathematica*, 25:379–387, 1935.

[10] V. Kreinovich, H. T. Nguyen, S. Ferson, and L. Ginzburg. From computation with guaranteed intervals to computation with confidence intervals: A new application of fuzzy techniques. In *Annual Meeting of the North American Fuzzy Information Processing Society, Proceedings NAFIPS 2002*, pages 418–423, 2002.

[11] H. T. Nguyen. *An Introduction to Random Sets*. Chapman & Hall/CRC, Boca Raton, 2006.

[12] M. Oberguggenberger and W. Fellin. Reliability bounds through random sets: nonparametric methods and geotechnical applications. *Computers & Structures*, 86:1093–1101, 2008.

[13] M. Oberguggenberger, J. King, and B. Schmelzer. Imprecise probability methods for sensitivity analysis in engineering. In G. de Cooman, J. Vejnarova, and M. Zaffalon, editors, *ISIPTA'07, Proceedings of the Fifth International Symposium on Imprecise Probability: Theories and Applications*, pages 317–326, Prague, 2007. Action M Agency, SIPTA.

[14] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.