Multi-criteria decision making with a special type of information about importance of groups of criteria

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Abstract

An axiomatic approach for solving a multi-criteria decision making problem is studied in the paper, which generally allows reducing a set of Pareto optimal solutions. The information about criteria in the problem is represented as the decision maker judgments of a special type. The judgments have a clear behavior interpretation and can be used in various decision problems. It is shown in the paper how to combine the judgments and to use them for reducing the Pareto set when they are provided by several decision makers. Two global criteria of decision making are introduced for comparing of decision alternatives. The first criterion based on the lower expectation, the second one is based on determining the belief and plausibility functions in the framework of Dempster-Shafer theory and uses the "threshold" probability for the final decision making. The numerical examples illustrate the proposed approach.

Keywords. Multi-criteria decision making, desirable gambles, Dempster-Shafer theory, judgments, preferences, Pareto set

1 Introduction

Most methods of *multi-criteria decision making* (MCDM) problems are somehow or other based on combining or aggregating *criteria*. According to these methods, *decision alternatives* (DA's) are compared by using the aggregated criterion. There are different ways for criteria combining. The widely-spread ways are linear, multiplicative and maximin combinations [3, 8]. For instance, the well-known analytic hierarchy process method proposed by Saaty [8] is based on the linear combination of criteria. However, in spite of the popularity of the aggregation methods for solving MCDM problems, they are ad hoc and have some justification related to certain applied areas. The main shortcoming of ad hoc methods is that it is often difficult to validate or to justify the optimal solutions. Another shortcoming is the necessity to have criteria with identical numerical scales.

Another part of methods is axiomatic, i.e., they are based on some axioms or properties and can be called strong methods. One of such the methods is reducing the so-called Pareto set of non-dominated solutions by utilizing some additional information about importance of criteria provided by experts, decision makers (DM's), etc. The amount of the additional information and its consistency determines the number of solutions in a reduced Pareto set. Ideally, the reduced Pareto set should consist of one solution.

Procedures for processing the additional information and for reducing the Pareto set totally depend on the type of available data or judgments. Many authors use the "weights" of criteria $\mathbf{v} = (v_1, ..., v_r)$ and different kinds of their ranking. For instance, Park and Kim [7], Kim and Ahn [4] distinguish between the following approaches to the elicitation of attribute weights: weak ranking ($v_i \geq v_j$); strict ranking ($v_i - v_j \geq \lambda_i$); ranking with multiples ($v_i \geq \lambda_i v_j$); interval form: ($\lambda_i \leq v_i \leq \lambda_i + \epsilon_i$); ranking of differences ($v_i - v_j \geq v_k - v_l$). Here $\lambda_i \geq 0$, $\epsilon_i \geq 0$.

Another very interesting type of judgments elicited from DM's or experts has been proposed by Noghin [5, 6] for reducing the Pareto optimal set in the framework of his theory of relative importance of criteria. Some details of the theory will be considered below. This type of judgments does not require to have identical numerical scales for criteria. It has a simple and clear behavior interpretation. Moreover, it turns out that many statements of the theory have analogues in the framework of desirable gambles [13, 14]. Therefore, MCDM problems in the framework of desirable gambles by relying on the Noghin's theory of relative importance of criteria are studied in the paper.

An interesting approach for eliciting the additional judgments from DM's or experts in MCDM problems (called the *DS/AHP method*) has been proposed by

Beynon *et al* [1, 2]. This method uses *Dempster-Shafer theory* in a framework of the analytic hierarchy process and allows to compare not only single DA's, but also groups of DA's. Beynon *et al* proposed to compare DA's, not criteria. However, a similar elicitation procedure can be applied to criteria [12]. Nevertheless, this method is also ad hoc and uses in the long run the linear aggregation (the analytic hierarchy process). Therefore, an attempt to modify it for reducing the Pareto optimal set in the framework of desirable gambles and Noghin's theory is made in the paper.

In the paper, "interval-valued" judgments as the extension of judgments proposed by Beynon *et al* are analyzed. These judgments have the following form: "I do not know which criterion is the most important, but this criterion belongs to the subset B of the set of criteria". Then these simple judgments are generalized to a more complex form, for instance, "I'm willing to pay w_i for the i-th criterion in order to get w_j for the j-th criterion. I'm also willing to pay w_k for the k-th criterion in order to get w_l for the l-th criterion. However, I do not know what is better." Their analysis is the next task for solving in the paper.

At the same time, we have to note that these judgments can be provided by several DM's. Therefore, the following problem to be solved is to combine them by taking into account the quality or weights of DM's. This will be done by introducing two global criteria of decision making, which are based on some statements of the Dempster-Shafer theory. In fact, these criteria can be regarded in the framework of second-order models [11]. It will be shown in the paper how to reduce the Pareto set of optimal solutions by applying the given information in the above forms and by using two proposed criteria.

The paper is organized as follows. The main definitions of MCDM are provided in Section 2. Some statements of Noghin's theory of relative importance of criteria are considered in Section 3. Noghin's theory is formulated in the framework of desirable gambles in Section 4. Different types of "interval-valued" judgments about criteria and their use for reducing a Pareto set are studied in Section 5. An illustrative numerical example is considered in the same section.

2 The MCDM problem statement

A general MCDM problem can be formulated in the following way. Suppose that there is a set of DA's $\mathbb{X} = \{X_1, ..., X_n\}$ consisting of n elements. Moreover, there is a set of criteria $\mathbb{C} = \{C_1, ..., C_r\}$ consisting of r elements, $r \geq 2$. For every DA, say the k-th DA, we can write the value of the i-th criterion $C_i(X_k)$ briefly denoted x_{ki}, $k = 1, ..., n$, $i = 1, ..., r$. Below, we will say that the i-th DA is characterized by the vector $X_i = (x_{i1}, ..., x_{ir})$. We will assume that the number of criteria and the number of DA's are finite.

To solve a MCDM problem is to find a set of all optimal solutions denoted by $\mathrm{Opt}\mathbb{X} \subseteq \mathbb{X}$, which can be regarded as the best solutions under certain conditions.

By making decisions, we usually have to take many objectives or criteria into account. The main feature here is that the different objectives are most likely conflicting and the final decision is commonly called a trade-off. When dealing with multiple objectives, solutions can be incomparable since they can dominate each other in different objectives. This lead to the notion of *Pareto optimality*, which is based on a partial order among the solutions. A solution is called Pareto optimal, if it is not dominated by any other solution, that is, if there is no other solution that is better in at least one objective and not worse in any of the other objectives. Naturally, Pareto optimal solutions are the candidates for a trade-off.

Let us give some standard definitions related to Pareto optimal solutions under assumption that there is no information about importance of criteria.

Definition 1 *$X \in \mathbb{X}$ dominates $Y \in \mathbb{X}$, denoted $X \succ Y$ iff $\forall i = 1, ..., r$, $x_i \geq y_i$ with at least one strict inequality.*

Definition 2 *$Y \in \mathbb{X}$ is a Pareto optimal alternative, also called an efficient alternative, iff $\nexists X \in \mathbb{X}$ such that $X \succ Y$. The set of all Pareto optimal alternatives in $\mathbb{X}$ or Pareto set is denoted $\mathcal{P}(\mathbb{X})$.*

It follows from the above definitions that the following inclusions are valid $\mathrm{Opt}\mathbb{X} \subseteq \mathcal{P}(\mathbb{X}) \subseteq \mathbb{X}$.

For many optimization problems, the number of Pareto optimal solutions can be rather large. Therefore, the problem of reducing Pareto optimal sets by obtaining the additional information is very important.

3 Noghin's relative importance of criteria

For reducing the Pareto optimal set, Noghin in [5] proposed the so-called theory of relative importance of criteria. This theory is based on the standard axioms and definitions of Pareto optimal solutions and the following additional axiom.

Axiom 1 *The preference relation $\succ$ is invariant with* *respect to positive linear transformation*[1].

The main idea of Noghin's theory is to compare criteria by means of parameters.

Definition 3 *Let $i, j \in N = \{1, 2, ..., r\}$, $i \neq j$. We say that the i-th criterion is more important than the j-th criterion with two positive parameters w_i and w_j if for any two vectors $X, Y \in \mathbb{X}$ such that*

$$x_i > y_i,\ x_j < y_j,\ x_k = y_k,\ \forall k \in N \backslash \{i, j\},$$

$$x_i - y_i = w_i,\ x_j - y_j = -w_j,$$

the relationship $X \succ Y$ is valid.

A behavior interpretation of the parameters w_i and w_j is the following. The DM is willing to pay w_j units for the j-th criterion in order to get w_i units for the i-th criterion. The relative importance coefficient is defined as

$$\theta_{ij} = \frac{w_j}{w_i + w_j}.$$

It can be seen that $0 < \theta_{ij} < 1$. At that, θ_{ij} is close to 1 if $w_j \gg w_i$. Moreover, θ_{ij} is close to 0 if $w_j \ll w_i$.

Introduce the following vector

$$W_{ij} = (0, ..., 0, w_i, 0, ..., -w_j, 0, ..., 0),$$

whose $r - 2$ elements are zero, the i-th element is w_i, the j-th element is $-w_j$. If the relation $X \succ Y$ is valid with the given parameters w_i and w_j, then we can write that the relation $W_{ij} \succ 0_r$ is valid. Here 0_r is the vector of r zero elements. The relation $W_{ij} \succ 0_r$ is equivalent to the relation $\Theta_{ij} \succ 0_r$, where

$$\Theta_{ij} = (0, ..., 0, 1 - \theta_{ij}, 0, ..., -\theta_{ij}, 0, ..., 0).$$

One of the main results of Noghin's theory of the relative importance of criteria is the following theorem.

Theorem 1 (Noghin [5]) *Let the i-th criterion be more important than the j-th criterion with the pair of positive parameters w_i and w_j. Then for any nonempty set of optimal vectors $\mathrm{Opt}\mathbb{X}$, it follows that*

$$\mathrm{Opt}\mathbb{X} \subseteq \mathcal{P}^*(\mathbb{X}) \subseteq \mathcal{P}(\mathbb{X}),$$

where $\mathcal{P}(\mathbb{X})$ is a set of Pareto-optimal vectors with respect to criteria $\mathbb{C} = \{C_1, ..., C_r\}$; $\mathcal{P}^*(\mathbb{X})$ is a set of Pareto-optimal vectors with respect to criteria $\mathbb{C}^* = \{C_1^*, ..., C_r^*\}$ such that

$$C_j^* = w_j C_i + w_i C_j,\ C_k^* = C_k,\ k \neq j.$$

[1] A binary relation $\mathcal{R}$ defined on $\mathbb{R}^r$ is said to be invariant with respect to positive linear transformation if for any vectors $X, Y, c \in \mathbb{R}^r$ and each positive number α the relationship $X \mathcal{R} Y$ implies $(\alpha X + c)\, \mathcal{R}\, (\alpha Y + c)$.

In other words, Theorem 1 provides a simple computation way for reducing the Pareto optimal set $\mathcal{P}(\mathbb{X})$. Its proof is based on properties of convex cones [6] produced by preferences of the form $W_{ij} \succ 0$. Theorem 1 is very important because it is a tool for dealing with the information about the relative importance of criteria. It can be easy written in terms of the relative importance coefficients θ_{ij}.

4 Sets of desirable gambles

A goal of this section is to consider Noghin's theory of the relative importance of criteria in the framework of desirable gambles [13, 14] and to show that its results and statements can be rather simply obtained on the basis of the framework. Preliminaries of the framework of desirable gambles given below can be found in [14].

Let Ω denote the set of possible outcomes under consideration. A bounded mapping from Ω to $\mathbb{R}$ (the real numbers) is called a *gamble*. Let $\mathcal{L}$ be a nonempty set of gambles. A mapping $\underline{P} : \mathcal{L} \to \mathbb{R}$ is called a lower prevision or lower expectation. The lower prevision of a gamble X is interpreted as a supremum buying price for X, meaning that it is acceptable to pay any price smaller than $\underline{P}(X)$ for the uncertain reward X. A lower prevision is said to be coherent when it is the lower envelope of some set of linear expectations, i.e., when there is a nonempty set of probability measures, $\mathcal{M}$, such that $\underline{P}(X) = \inf \{E_P(X) : P \in \mathcal{M}\}$ for all $X \in \mathcal{L}$, where $E_P(X)$ denotes the expectation of X with respect to P. The conjugate upper prevision is determined by $\overline{P}(X) = -\underline{P}(-X)$. It is interpreted as an infimum selling price for X.

For $X, Y \in \mathcal{L}$, write $X \geq Y$ to mean that $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$, and write $X > Y$ to mean $X \geq Y$ and $X(\omega) > Y(\omega)$ for some $\omega \in \Omega$. According to Walley [13], a gamble X is *inadmissible* in $\mathcal{L}$ when there is $Y \in \mathcal{L}$ such that $Y \geq X$ and $Y \neq X$. Otherwise X is *admissible* in $\mathcal{L}$. The subset $\mathcal{P}$ of admissible gambles in $\mathcal{L}$ is an analogue of the Pareto set in MCDM. A set of desirable gambles, denoted by $\mathcal{D}$, is a subset of $\mathcal{L}$. A set of desirable gambles is said to be *coherent* when it satisfies the four axioms:

D1. $0 \notin \mathcal{D}$.

D2. if $X \in \mathcal{L}$ and $X > 0$, then $X \in \mathcal{D}$.

D3. if $X \in \mathcal{D}$ and $c \in \mathbb{R}_+$, then $cX \in \mathcal{D}$.

D4. if $X \in \mathcal{D}$ and $Y \in \mathcal{D}$, then $X + Y \in \mathcal{D}$.

Thus a coherent set of desirable gambles is a convex cone of gambles that contains all positive gambles ($X > 0$) but not the zero gamble. Consequence of the axioms: If $X \in \mathcal{D}$ and $Y \geq X$, then $Y \in \mathcal{D}$.

It can be seen from the axioms of coherence that D3 and D4 coincide with Axiom 1 about positive linear transformation used by Noghin in his theory. Moreover, it can be seen from Definition 3 that assessments of the parameters w_i and w_j can be regarded as some extension of the probability ratios studied by Walley [13]. The probability ratios generalize the comparative probability judgments and have the form "A is at least l times as probable as B", where l is a positive number. The gamble $A-lB$ is almost desirable. This implies that $A \succ lB$.

Walley states that there is a one-to-one correspondence between coherent sets of desirable gambles and coherent partial preference orderings, defined by $X \succ Y$ if and only if $X - Y \in \mathcal{D}$. This is very important statement which allows to find the same correspondence between the framework of desirable gambles and Noghin's theory.

If a closed convex set of probability measures $\mathcal{M}$ is given, then we can define a set of desirable gambles as follows:

$$\mathcal{D} = \{X \in \mathcal{L} : X > 0 \text{ or } \mathbb{E}_P(X) > 0, \forall P \in \mathcal{M}\}. \quad (1)$$

Then $\mathcal{D}$ is coherent and $\mathcal{M}$ can be recovered from it by

$$\mathcal{M} = \{P : \mathbb{E}_P(X) \geq 0,\ \forall X \in \mathcal{D}\}. \quad (2)$$

Note that (1) can be rewritten as

$$\mathcal{D} = \{X \in \mathcal{L} : X > 0 \text{ or } \underline{\mathbb{E}}_{\mathcal{M}}(X) > 0\}. \quad (3)$$

Suppose that we have information about the relative importance of the i-th and the j-th criteria, i.e. the i-th criterion is more important than the j-th criterion with two positive parameters w_i and w_j. Let us return to the vector W_{ij} produced by the parameters w_i, w_j and consider again the relation $W_{ij} \succ 0_r$ (see Section 3). This relation can be written in the framework of desirable gambles as the condition $W_{ij} - 0_r \in \mathcal{D}$ or just $W_{ij} \in \mathcal{D}$. In other words, the information about the relative importance of the i-th and the j-th criteria can be represented as the condition that the vector W_{ij} belongs to the set of desirable gambles.

Now we reformulate Noghin's theorem and prove it in terms of desirable gambles.

Let X and Y be two DA's. We will denote below the vector $Z = X - Y$ and its components $z_k = x_k - y_k$ for short.

Theorem 2 *The preference $X \succ Y$ is valid if $X^* > Y^*$ and $W_{ij} \in \mathcal{D}$. Here $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ such that*

$$x_j^* = w_j x_i + w_i x_j,\ x_k^* = x_k,\ k \neq j,$$

$$y_j^* = w_j y_i + w_i y_j,\ y_k^* = y_k,\ k \neq j.$$

Proof. Note that $X \succ Y$ if $X - Y = Z \in \mathcal{D}$ or $\mathbb{E}_P(Z) > 0$ for all $P \in \mathcal{M}$. The condition $W_{ij} \in \mathcal{D}$ restricts the set $\mathcal{M}$ of possible probability measures by the constraint $\mathbb{E}_P(W_{ij}) \geq 0$. If we denote $P = (\pi_1, ..., \pi_r)$, then the above constraint can be rewritten as $w_i \pi_i - w_j \pi_j \geq 0$. This implies that the set of all probability measures $\mathcal{M}$ is reduced to the subset $\mathcal{M}(ij) \subseteq \mathcal{M}$. The subset $\mathcal{M}(ij)$ is defined by the constraints

$$\sum_{k=1}^{r} \pi_k = 1,\ \pi_k \geq 0,\ \forall k \in N,$$

$$w_i \pi_i - w_j \pi_j \geq 0.$$

Here $N = \{1, 2, ..., r\}$.

Let us find extreme points of $\mathcal{M}(ij)$. They are

$$(0, ..., 0, 1_k, 0, ..., 0),\ \forall k \in N \backslash \{j\},$$

and

$$\pi_i = \frac{w_j}{w_i + w_j},\ \pi_j = \frac{w_i}{w_i + w_j},$$

$$\pi_k = 0,\ \forall k \in N \backslash \{j\}.$$

The last extreme point is produced by the equality $w_i \pi_i - w_j \pi_j = 0$.

The extreme points define the set of probability distributions $\mathcal{M}(ij)$. Therefore, if we prove that the inequality $\mathbb{E}_P(Z) > 0$ is valid for extreme points, then this inequality will be valid for all $P \in \mathcal{M}(ij)$. The first $k-2$ extreme points give

$$\mathbb{E}_P(Z) = z_k,\ \forall k \in N \backslash \{i, j\}.$$

The last extreme point gives

$$\mathbb{E}_P(Z) = \pi_i z_i + \pi_j z_j = \frac{w_j z_i}{w_i + w_j} + \frac{w_i z_j}{w_i + w_j}.$$

At the same time, the condition $X^* > Y^*$ implies that $z_k > 0$ or $z_k = 0$ for all $k \neq j$, and $w_j z_i + w_i z_j > 0$. Hence $\mathbb{E}_P(Z) > 0$ for all $P \in \mathcal{M}(ij)$ and $X \succ Y$, as was to be proved. ■

Example 1 *Consider the simplified and modified example of the optimal choice of a place for the airport construction given by Keeney and Raiffa in their book [3] and solve it by using Noghin's theory. The problem is to decide where a new airport should be constructed in accordance with the following criteria[2]: minimize investment of capital in million dollars (C_1), maximize carrying capacity in the daily number of air travellers (C_2), maximize safety expressed in the 9-point*

[2]The example is given with some changes.

scale from 1 *till* 9 *(*C_3*), minimize remoteness in kilometers (*C_4*). There are four places for the construction (DA's) denoted* X_1*,* X_2*,* X_3*,* X_4*. The MCDM problem can also be represented by means of the matrix*

	C_1	C_2	C_3	C_4
X_1	-20	15000	6	-3
X_2	-30	25000	4	-1
X_3	-40	18000	7	-5
X_4	-25	20000	5	-2

.

Here negative values are taken in order to replace the "minimization" goals by the "maximization" ones.

It can be seen from the matrix that all the DA's belong to the Pareto set.

The DM is willing to pay $w_3 = 3$ *units for the third criteria in order to get* $w_1 = 2$ *units for the first criterion. The provided information can be represented by the gamble* $W_{13} = (2, 0, -3, 0) \in \mathcal{D}$ *or equivalently by the gamble* $\Theta_{13} = (2/5, 0, -3/5, 0) \in \mathcal{D}$*.*

Then we write the modified matrix by using Noghin's theorem

	C_1	C_2	$3 \cdot C_1 + 2 \cdot C_3$	C_4
X_1	-20	15000	-48	-3
X_2	-30	25000	-82	-1
X_3	-40	18000	-106	-5
X_4	-25	20000	-65	-2

.

Hence, we reduce the Pareto set which now consists of three DA's X_1*,* X_2 *and* X_4*. It can be seen that it is not enough to have the supplied judgment for getting one optimal solution.*

5 Groups of the most important or preferable criteria

Let us quickly return to the analytic hierarchy process method. In addition to the fact that it must perform very complicated and numerous pairwise comparisons amongst alternatives the method uses precise estimates of experts or DM's. This condition can not be satisfied in many applications because judgments elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments. In order to overcome these difficulties and to extend the analytic hierarchy process on a more real elicitation procedures, Beynon *et al* [1, 2] proposed a method using Dempster-Shafer theory and called the DS/AHP method. The method was developed for decision making problems with a single DM, and it applies the analytic hierarchy process method for collecting the preferences from the DM and for modelling the problem as a hierarchical decision tree. It should be noted that the main idea underlying the DS/AHP method is not applying Dempster-Shafer theory to the analytic hierarchy process method. It is comparison of groups of alternatives with a whole set of alternatives. Such the type of comparison is equivalent to the preferences stated by the DM. In other words, Beynon *et al* [1, 2] proposed to consider preferences of the form $B \succ \mathbb{X}$ with some degree v of it, where B is a subset or a group of DA's, $\mathbb{X}$ is the set of all alternatives, v is a positive number in accordance with some scale [8]. The same can be carried out for the criteria, i.e., we can consider preferences of the form $D \succ \mathbb{C}$, where D is a subset of criteria. It is obvious that this preference can be rewritten in the form $D \succ \mathbb{C}\backslash D$. The authors of the papers [1, 2] assign to every subset B some basic probability assignment (BPA) [9] denoted $m(B)$. The same can be done for criteria.

Such the elicitation procedure has some virtues. First, a DM does not need to choose the most important criterion from the set of criteria. The DM chooses a subset of criteria by assuming that one of these criteria is the most important or important with some degree of importance. However, these judgments are used in the aforementioned aggregating criteria methods which are ad hoc. As a result, it is difficult to validate the approach in specific applied problems.

Now we will formalize the above elicitation procedure in the framework of Noghin's theory and desirable gambles. Then we will study how this procedure can be applied to reducing the set of Pareto optimal solutions.

Suppose that there is a set of t judgments of the form $D_l \succ \mathbb{C}$ with the corresponding BPA's $m(D_l)$, $l = 1, ..., t$. The first question is to construct a criterion (criteria) for the validity of the preference $X \succ Y$. These criteria will be called global in order to distinguish them from the criteria $C_1, ..., C_r$ of the considered MCDM problem.

The second question is the computation rules for the validity of $X \succ Y$.

5.1 Simple comparison judgments

First we consider simple comparison judgments of the form: "I do not know which criterion is the most important, but this criterion belongs to the subset $B \subseteq \mathbb{C}$". Here the degree v is assumed to be unknown. Suppose that the unknown important criterion has the number k and the subset B contains t elements with numbers from the index set[3] B^0. Denote $B^1 = N\backslash B^0$, $N = \{1, ..., r\}$. Then we can provide

[3] The set of indices of elements of B will be denoted B^0. The set of indices of elements of $\mathbb{C}\backslash B$ will be denoted B^1.

$r-t$ judgments:

"The k-th criterion is more important than the j-th criterion from $\mathbb{C}\backslash B$ with the pair of positive parameters $w_k = 1$ and $w_j = 1$".

Here $k \in B^0$ and $j \in B^1$. So, every judgment produces the gamble[4]

$$W_{kj} = (0, ..., 0, 1_k, 0, ..., -1_j, 0, ..., 0), \qquad (4)$$

such that $W_{kj} \succ 0_r$, $k \in B^0$, $j \in B^1$.

It should be noted that the simple comparison judgment with the above desirable gamble W_{kj} can be applied to decision problems with uniform criteria, i.e., criteria have identical numerical scales.

Now we can find the subset $\mathcal{M}(k, B^1) \subseteq \mathcal{M}$ of probability distributions $P = (\pi_1, ..., \pi_r)$ restricted by the desirable gambles W_{kj}, $j \in B^1$, or equivalently its extreme points. The subset $\mathcal{M}(k, B^1)$ is produced by the judgment about comparison of the k-th criterion and the j-th criterion.

Proposition 1 *Given the additional information in the form (4), the preference $X \succ Y$ is valid if the condition*

$$z_k + \sum_{j \in L} z_j \geq 0$$

is valid for all $L \subseteq B^1$ and $z_i \geq 0$ for all $i \in B^0$.

Proof. Let us find the subset $\mathcal{M}(k, B^1)$. It follows from (2) and from (4) that this set is produced by the constraints[5]

$$\pi_k - \pi_j \geq 0, \ j \in B^1, \ \pi_i \geq 0, \ i \in N,$$

$$\pi_1 + \pi_2 + ... + \pi_r = 1.$$

Consider r equalities instead of inequalities in the above constraints. Hence, we get extreme points of the form:

$$\begin{aligned}
&\pi_k = 1, \ \pi_i = 0, \ \forall i \neq k, \\
&\pi_k = 1/2, \ \pi_{j_1} = 1/2, \ j_1 \in B^0, \\
&\pi_k = 1/3, \ \pi_{j_1} = \pi_{j_2} = 1/3, \ j_1, j_2 \in B^0, \\
&\quad \cdots \\
&\pi_k = 1/(r-t+1), \ \pi_{j_i} = 1/(r-t+1), \\
&j_i, \in B^0, \ i = 1, ..., r-t.
\end{aligned}$$

Only non-zero elements of extreme points are written here. The proof directly follows from the condition of desirability of gambles $X - Y$, which is of the form: $\mathbb{E}_P(X - Y) \geq 0$, $\forall P \in \mathrm{extr}(\mathcal{M}(k, B^1))$. ■

Several conditions in Proposition 1 can be replaced by one equivalent condition

$$z_k + \min_{L \subseteq B^1} \sum_{j \in L} z_j \geq 0. \qquad (5)$$

So, the Pareto set can be reduced by using condition (5) for every pair of DA's.

It also follows from the proof of Proposition 1 and from (5) that the lower expectation of the gamble $Z = X - Y$ under conditions $W_{kj} \succ 0_r$, $j \in D_l^1$, denoted $\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(Z)$ is of the form

$$\begin{aligned}
\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(Z) &= \min_{L \subseteq D_l^1} \mathbb{E}_P(Z) \\
&= \min_{L \subseteq D_l^1} \frac{1}{q_L + 1} \left(z_k + \sum_{j \in L} z_j \right). \qquad (6)
\end{aligned}$$

Here L is a subset of D_l^1; q_L is the number of elements in L ($q_L = \mathrm{card}\,(L)$).

We have considered how to formalize "one-side interval" preference[6]. However, the additional information about BPA's of the corresponding "intervals" has not been applied to the studied MCDM problem. In order to take this additional information into account, we have to introduce the so-called *global criteria* which establish how to compare two DA's from the Pareto set in accordance with all the available information. It should be noted that the global criteria differ from the criteria (goals) $C_1, ..., C_r$.

Below two global criteria for comparison DA's X and Y are proposed.

5.1.1 The first global criterion

The first global criterion is based on the definition of the desirability (3) and can be written as follows. The preference $X \succ Y$ is valid if $\underline{\mathbb{E}}_{\mathcal{P}} \mathbb{E}_P(X - Y) > 0$. Here $\mathcal{P}$ is a set of probability distributions defined on the partition of $\mathcal{M}$ produced by the given information in the form of preferences $D_l \succ \mathbb{C}$ with BPA's $m(D_l)$, $l = 1, ..., t$. For computing the lower expectation, we can use the approach introduced by Strat [10], which directly relies on belief functions based on some basic probability assignment $m(\cdot)$. According to this approach, the lower expectation of $\underline{\mathbb{E}}h$ of a function h is

[4] The reason why the parameters $w_k = 1$ and $w_j = 1$ are taken for formalizing the simple comparison judgments is clearly seen from the proof of Proposition 1.

[5] One can see from the first $r-t$ constraints that they correspond to the comparison of probabilities π_k and π_j, i.e., they formalize the judgment "the k-criterion is as probable as j-th criterion". This implies that the parameters $w_k = 1$ and $w_j = 1$ form the simple comparison.

[6] We have still studied judgments with a fixed k and "interval" $\mathbb{C}\backslash D$ without analyzing the interval D.

determined as follows:

$$\underline{\mathbb{E}}h = \sum_{l=1}^{t} m(D_l) \min_{x \in D_l} h(x).$$

Let $\mathcal{M}(k, D_l^1)$ be a subset of probability distributions produced by conditions $W_{kj} \succ 0_r$, $j \in D_l^1$. Then $m(D_l)$ corresponds to the union of subsets

$$\mathcal{M}(D_l) = \cup_{k \in D_l^0} \mathcal{M}(k, D_l^1).$$

Then the function $h(x)$ in the considered case is the expectation $\sum_{i=1}^{r} \pi_i z_i$. Hence, we get

$$\underline{\mathbb{E}}_{\mathcal{P}} \mathbb{E}_P(Z) = \sum_{l=1}^{t} m(D_l) \left(\min_{k \in D_l^0} \inf_{P \in \mathcal{M}(D_l)} \sum_{i=1}^{r} \pi_i z_i \right).$$

However, there holds

$$\inf_{P \in \mathcal{M}(D_l)} \sum_{i=1}^{r} \pi_i z_i = \underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y).$$

Hence, Proposition 2 can be stated from the above reasoning.

Proposition 2 *Suppose that there is a set of t judgments of the form $D_l \succ \mathbb{C}$ with the corresponding BPA's $m(D_l)$, $l = 1, ..., t$. The preference $X \succ Y$ is valid in accordance with the first global criterion if the condition*

$$\begin{aligned} &\underline{\mathbb{E}}_{\mathcal{P}} \mathbb{E}_P(X - Y) \\ &= \sum_{l=1}^{t} m(D_l) \min_{k \in D_l^0} \underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y) \geq 0 \end{aligned}$$

is valid. Here $\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y)$ is defined from (6).

5.1.2 The second global criterion

The second criterion is based on the definition of belief and plausibility functions. According to this criterion, we can say about the preference $X \succ Y$ with some "threshold" or confident probability which lies between the belief and plausibility functions. Note that the set $\mathcal{M}$ of all probability distributions can be divided into two subsets $\mathcal{M}_1$ and $\mathcal{M}_2$. The subset $\mathcal{M}_1$ satisfies the condition $X - Y \in \mathcal{D}$. The subset $\mathcal{M}_2$ satisfies the condition $X - Y \notin \mathcal{D}$. Then all subsets $\mathcal{M}(D_l)$ belonging to $\mathcal{M}_1$ form the belief function $\mathrm{Bel}\,(X - Y \in \mathcal{D})$. Note that the subset $\mathcal{M}(D_l)$ intersects $\mathcal{M}_1$ if at least for one of the values k from D_l^0 the subset $\mathcal{M}(k, D_l^1)$ belongs to $\mathcal{M}_1$. The proposition follows from the above.

Proposition 3 *Suppose that there is a set of t judgments of the form $D_l \succ \mathbb{C}$ with the corresponding BPA's $m(D_l)$, $l = 1, ..., t$. The preference $X \succ Y$ is valid in accordance with the second global criterion with a probability belonging to the interval with the following bounds*

$$\mathrm{Bel}\,(X - Y \in \mathcal{D}) = \sum_{l \in R} m(D_l),$$

$$\mathrm{Pl}\,(X - Y \in \mathcal{D}) = \sum_{l \in G} m(D_l),$$

where R is a set of indices such that for every $l \in R$, there holds

$$\min_{k \in D_l^0} \underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y) > 0,$$

G is a set of indices such that for every $l \in G$, there holds

$$\max_{k \in D_l^0} \underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y) > 0.$$

Here $\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y)$ is defined from (6).

The belief function is the lower (pessimistic or conservative) bound for the probability of the preference $X \succ Y$.

Note that Propositions 2 and 3 are rather general and their main results do not depend on the way of obtaining the lower expectation $\underline{\mathbb{E}}_{\mathcal{M}(k, D_l^1)}(X - Y)$. This implies that the propositions can be generalized by studying a more practical case when we have parameters of the criteria importance w_k and w_j (see Section 3).

5.2 General case

In this section, we generalize the simple comparison judgments by introducing parameters for every pair of criteria, i.e. for every k, DM's supply different parameters $w_j^{(k)}$ for all $j \in B^1$. This is a possible formalization of judgments: "The k-th criterion from B is more important than the j-th criterion from $\mathbb{C} \backslash B$ with the pair of positive parameters w_k and w_j". A special case of the above judgment is the preferences provided by DM's with some degree v under condition that the criteria have identical scales. In this case, we have $v = w_j / w_k$ or $v = \theta_{kj} / (1 - \theta_{kj})$. However, we consider the general case.

Assume for example that $\mathbb{C} = \{C_1, C_2, C_3\}$, $B = \{C_1, C_2\}$, and $\mathbb{C} \backslash B = \{C_3\}$. Then the corresponding judgment of a DM might also have the form: "I'm willing to pay w_3 for C_3 in order to get w_1 for C_1. I'm also willing to pay w_3 for C_3 in order to get w_2 for C_2. However, I do not know what is better. "

Suppose that we have a set of judgments such that every judgment produces the gamble

$$W_{kj} = (0, ..., 0, w_k, 0, ..., -w_j, 0, ..., 0), \qquad (7)$$

such that $W_{kj} \succ 0_r$, $k \in B^0$, $j \in B^1$.

Now we can find the set $\mathcal{M}$ restricted by the desirability of gambles W_{kj} or equivalently its extreme points.

Proposition 4 *Given the additional information in the form (7), the preference $X \succ Y$ is valid if the condition*

$$z_k + \sum_{j \in L} \frac{w_k}{w_j} z_j \geq 0$$

is valid for all $L \subseteq B^1$ and $z_i \geq 0$ for all $i \in B^0$.

Proof. Denote $v_{kj} = w_k/w_j$. It follows from (2) and from (7) that the set $\mathcal{M}$ is produced by the constraints

$$v_{kj}\pi_k - \pi_j \geq 0, \ j \in B^1,$$

$$\pi_i \geq 0, \ i \in N,$$

$$\pi_1 + \pi_2 + ... + \pi_r = 1.$$

Case 1. $v_{kj}\pi_k = \pi_j$, $\pi_i = 0$, $\forall i \in N \backslash \{k, j\}$. Then for every $j \in B^1$, we get the extreme points

$$\pi_k = \frac{1}{1 + v_{kj}}, \ \pi_j = \frac{v_{kj}}{1 + v_{kj}},$$
$$\pi_i = 0, \forall i = N \backslash \{k, j\}.$$

Case 2. $v_{kj_1}\pi_k = \pi_{j_1}$, $v_{kj_2}\pi_k = \pi_{j_2}$, $\pi_i = 0$, $\forall i \in N \backslash \{k, j_1, j_1\}$. Then for every pair $j_1, j_2 \in B^1$, we get the extreme points

$$\pi_k = \frac{1}{1 + v_{kj_1} + v_{kj_2}}, \ \pi_{j_1} = \frac{v_{kj_1}}{1 + v_{kj_1} + v_{kj_2}},$$
$$\pi_{j_2} = \frac{v_{kj_2}}{1 + v_{kj_1} + v_{kj_2}}, \ \pi_i = 0, \forall i = N \backslash \{k, j_1, j_2\}.$$

By continuing the analysis of the cases, we write the following last case.

Case $r - t + 1$. $v_{kj_i}\pi_k = \pi_{j_i}$, $i = 1, ..., r - t$, $\pi_l = 0$, $\forall l \in B^0 \backslash \{k\}$. Then we get the extreme points

$$\pi_k = \frac{1}{1 + \sum_{i=1}^{r-t} v_{kj_i}},$$
$$\pi_{j_i} = \frac{v_{kj_i}}{1 + \sum_{i=1}^{r-t} v_{kj_i}}, \ i = 1, ..., r - t,$$
$$\pi_l = 0, \ \forall l \in B^0 \backslash \{k\}.$$

The proof directly follows from the condition of desirability of the gamble $Z = X - Y$, which is of the form: $\mathbb{E}_P(Z) \geq 0$, $\forall P \in \text{extr}(\mathcal{M})$. Hence, for every subset $L \subseteq B^1$, we can write the expectations as follows:

$$\mathbb{E}_P(Z) = \frac{z_k}{1 + \sum_{i \in L} v_{ki}} + \sum_{j \in L} \frac{v_{kj} z_j}{1 + \sum_{i \in L} v_{ki}}.$$

Since $v_{kj_i} \geq 0$ for all k, j, i, then $\mathbb{E}_P(Z) \geq 0$ for every extreme point if

$$z_k + \sum_{j \in L} v_{kj} z_j \geq 0, \ L \subseteq B^1,$$

as was to be proved. ∎

We get the rather simple expressions for reducing the Pareto set.

Generally speaking, the values w_k in Proposition 4 may be different for different values of $j \in L$ and the index k_j should be used. However, we assume for simplicity that the parameters w_k are identical for every W_{kj}. Moreover, it can be seen from the proof of Proposition 4 that the condition of the preference $X \succ Y$ depends only on the ratio w_k/w_j and we can always change w_k and w_j without changing the above ratio.

Let us considers a special case when each of the subsets B^1 and B^0 consists of one element.

Corollary 1 *Suppose that $B^1 = \{k\}$ and $B^0 = \{j\}$. Then the preference $X \succ Y$ is valid if the conditions*

$$w_j z_k + w_k z_j \geq 0, \ z_i \geq 0, \ \forall i \neq j,$$

are valid.

One can see that the conditions in Corollary 1 coincide with the conditions in Theorems 1 and 2.

Several conditions in Proposition 4 can be replaced by one equivalent condition

$$z_k + w_k \min_{L \subseteq B^1} \sum_{j \in L} z_j w_j^{-1} \geq 0. \qquad (8)$$

By using (8) and Propositions 2, 3 we can write the following corollary.

Corollary 2 *If there are judgments of one DM ($l = 1$, $D_1 = D$) with the BPA $m(D_l) = 1$, then the preference $X \succ Y$ is valid in accordance with the first global criterion if the conditions*

$$\min_{k \in D^0} \{z_k + T w_k\} \geq 0$$

are valid. Here

$$T = \min_{L \subseteq D^1} \sum_{j \in L} z_j w_j^{-1}.$$

Moreover, the belief function $\mathrm{Bel}(X-Y\in\mathcal{D})$ *is 1 in accordance with the second global criterion if the above conditions are valid.*

It also follows from the proof of Proposition 4 and from (8) that the lower expectation of the gamble $Z=X-Y$ under conditions $W_{kj}\succ 0_r$, $j\in D_l^1$, denoted $\underline{\mathbb{E}}_{\mathcal{M}(k,D_l^1)}(Z)$ is of the form

$$\begin{aligned}\underline{\mathbb{E}}_{\mathcal{M}(k,D_l^1)}(Z) &= \min_{L\subseteq D_l^1}\mathbb{E}_P(Z)\\ &= \min_{L\subseteq D_l^1}\frac{z_k+w_k\sum_{j\in L}z_jw_j^{-1}}{1+w_k\sum_{i\in L}w_i^{-1}}. \quad (9)\end{aligned}$$

Then Propositions 2 and 3 can be used in the considered case of the elicited information if we replace (6) by (9).

Example 2 *Let us return to Example 1. The judgment of the first DM is the following:*

"I'm willing to pay $w_2=15000$ *for* C_2 *in order to get* $w_1=15$ *for* C_1 *and I'm willing to pay* $w_4=7$ *for* C_4 *in order to get* $w_1=15$ *for* C_1*. I'm also willing to pay* $w_2=24000$ *for* C_2 *in order to get* $w_3=1$ *for* C_3 *and I'm willing to pay* $w_4=10$ *for* C_4 *in order to get* $w_3=1$ *for* C_3*. However, I do not know what is better. "*

The above judgment can be formalized as $D_1=\{C_1,C_3\}\succ\{C_2,C_4\}$*. The judgment of the second DM is the following:*

"I'm willing to pay $w_3=6$ *for* C_3 *in order to get* $w_1=30$ *for* C_1*. I'm also willing to pay* $w_3=8$ *for* C_3 *in order to get* $w_2=10000$ *for* C_4*. I'm also willing to pay* $w_3=20$ *for* C_3 *in order to get* $w_4=1$ *for* C_4*. However, I do not know what is better. "*

This judgment can be formalized as $D_2=\{C_1,C_2,C_4\}\succ\{C_3\}$*.*

The BPA of the first DM is $m(D_1)=0.6$*. The BPA of the second DM is* $m(D_2)=0.4$*.*

Let us find $\underline{\mathbb{E}}_{\mathcal{M}(k,D_l^1)}(X-Y)$*. If* $D_1^1=\{2,4\}$ *and* $k=1,3$*, then it follows from (9) that*

$$\begin{aligned}\underline{\mathbb{E}}_{\mathcal{M}(1,D_1^1)} = \min\Bigg(& z_1, \frac{z_1+15z_2/15000}{1+15/15000},\\ & \frac{z_1+15z_4/7}{1+15/7},\\ & \frac{z_1+15z_2/15000+15z_4/7}{1+15/15000+15/7}\Bigg),\end{aligned}$$

Table 1: Comparison of DA's by using two criteria

$X\succ Y$	$\underline{\mathbb{E}}_{\mathcal{P}}\mathbb{E}_P(X-Y)$	Bel	Pl
$X_1\succ X_2$	-1199	0.6	1
$X_1\succ X_3$	0.14	0.4	1
$X_1\succ X_4$	0.09	0.6	1
$X_2\succ X_1$	-10	0	0.4
$X_3\succ X_1$	-1614	0	0.6
$X_4\succ X_1$	-1614	0	0.6
$X_2\succ X_3$	-2.13	0	1
$X_2\succ X_4$	-5	0	0.4
$X_3\succ X_2$	-2810	0	0.6
$X_3\succ X_4$	-810.2	0	0.6

$$\begin{aligned}\underline{\mathbb{E}}_{\mathcal{M}(3,D_1^1)} = \min\Bigg(& z_3, \frac{z_3+1z_2/24000}{1+1/24000},\\ & \frac{z_3+1z_4/10}{1+1/10},\\ & \frac{z_3+1z_2/24000+1z_4/10}{1+1/24000+1/10}\Bigg).\end{aligned}$$

If $D_2^1=\{3\}$ *and* $k=1,2,4$*, then it follows from (9) that*

$$\underline{\mathbb{E}}_{\mathcal{M}(1,D_2^1)} = \min\left(z_1, \frac{z_1+30z_3/6}{1+30/6}\right),$$

$$\underline{\mathbb{E}}_{\mathcal{M}(2,D_2^1)} = \min\left(z_2, \frac{z_2+10000z_3/8}{1+10000/8}\right),$$

$$\underline{\mathbb{E}}_{\mathcal{M}(4,D_2^1)} = \min\left(z_4, \frac{z_4+1z_3/20}{1+1/20}\right).$$

The computation results with using Propositions 2 and 3 are shown in Table 1. It can be seen from Table 1 that the reduced Pareto set ***in accordance with the first criterion*** $\underline{\mathbb{E}}_{\mathcal{P}}\mathbb{E}_P(X-Y)>0$ *of decision making consists of two DA's* X_1 *and* X_2 *because* $\underline{\mathbb{E}}_{\mathcal{P}}\mathbb{E}_P(X_1-X_3)=0.14>0$ *and* $\underline{\mathbb{E}}_{\mathcal{P}}\mathbb{E}_P(X_1-X_4)=0.09>0$*. However, by using* ***the second criterion*** *of decision making* ***with the "threshold" probability*** 0.6 *for the belief function, we can construct the reduced Pareto set consisting of two DA's* X_1 *and* X_3*.*

6 Conclusion

A method for solving a MCDM problem with the elicited information about criteria of a special form has been proposed in the paper. The main feature of the method is that it is based on reducing a set of Pareto optimal solutions and does not use aggregation of criteria for solving the problem. The additional information applied in the proposed method is rather natural because DM's or experts are usually able to provide parameters w_i and w_j whose simple behavior interpretation is considered in Section 3 and in Example 2.

It has been shown in the paper that Noghin's theory of relative importance of criteria can be easy represented in terms of sets of desirable gambles and many statements of the theory can be proved by means of desirable gambles and the imprecise probability theory.

Two global criteria of decision making are introduced. The first criterion based on the lower expectation uses the second-order models as a main tool for determining whether a preference $X \succ Y$ is valid or not. The second criterion is based on determining the belief and plausibility function in the framework of Dempster-Shafer theory. It uses the so-called "threshold" probability for the final decision making.

One can see from the proposed expressions (6), (9) and Propositions 2 and 3 that all the mathematical expressions are rather simple from the computation point of view and they do not require special procedures for computing the lower expectation $\underline{\mathbb{E}}_{\mathcal{P}}\mathbb{E}_P(X - Y)$ and the belief and plausibility functions.

Some specialists in Dempster-Shafer theory might object that the condition of independence of DM's in combining their judgments is not taken into account. Of course, we could assume that the DM's are independent and use, for instance, Dempster rule of combination. However, the main aim of the paper is to propose an approach for reducing the Pareto set on the basis of the special information, in particular, on judgments producing sets of gambles (7). Various modifications and features of the approach can be studied in further research.

It should be noted that the simple case has been studied in the paper when only judgments of the special type are provided by DM's. However, the proposed approach can be extended on a more complicated case. Therefore, a direction for further work is to investigate the general cases.

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