

On solutions of stochastic differential equations with parameters modelled by random sets

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Abstract

We consider ordinary stochastic differential equations whose coefficients depend on parameters. Conditions are given under which modelling the parameter uncertainty by compact-valued random sets leads to set-valued stochastic processes. Finally, we define analogues of first entrance times for set-valued processes.

Keywords. Stochastic differential equation, random set, set-valued stochastic process, first entrance time.

1 Introduction

Stochastic differential equations of the form

$$dx_t = f(t, x_t)dt + G(t, x_t)dw_t \quad (1)$$

or the equivalent integral form

$$x_t = x_{t_0} + \int_{t_0}^t f(s, x_s)ds + \int_{t_0}^t G(s, x_s)dw_s \quad (2)$$

with initial value x_{t_0} , coefficients $f : [t_0, \bar{t}] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $G : [t_0, \bar{t}] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ and $\{w_t\}_{t \in [t_0, \bar{t}]}$ being an m -dimensional Wiener process (Brownian motion) are used in many applications to model classical problems in physics and engineering under random disturbances. The theory of such equations and their solutions being stochastic processes can be found in [1] or [12], for example.

The motivation for this work is the desire for ultimately investigating mechanical systems under stochastic excitations depending on parameters. The purpose of this article is thus to consider SDEs whose initial value x_{t_0} and coefficients f and G depend on parameters. The uncertainty of these parameters can be modelled by random variables which requires the assumption of certain probability distributions. But in practice, there may only be scarce information available like a small sample size

or estimates on the mean value and the variance. Hence, the classical probabilistic approach might involve tacit assumptions that cannot be verified and the need for alternative uncertainty models may arise (for a general discussion see for example [24]). Among those alternative models are random sets which can be interpreted as imprecise observations of random variables, that is, instead of a single value one assigns a set which is supposed to include the actual value to each of the elements of the underlying probability space. It has been demonstrated in [23, 25, 26] how random intervals constructed from Tchebycheff's inequality can serve as a non-parametric model of the variability of a parameter, given its mean value and variance as sole information.

We will start in Section 2 with a rather detailed review of the basic theory of stochastic processes and measurability of random sets which is necessary to understand the definitions and propositions of Section 3 where conditions will be given under which solution processes continuously depend on the parameters contained in x_{t_0} , f and G . We will show that this continuity together with using random compact sets for modelling parameter uncertainty leads to set-valued processes with compact values which are continuous with respect to the Hausdorff metric. Section 4 discusses possible definitions of analogues of first entrance times for set-valued processes and their representability by first entrance times of selections. In Section 5 an example is given to illustrate the theoretical concept developed in the foregoing sections.

We point out that this article addresses the case where f and G are \mathbb{R}^d -valued coefficient functions depending on random set parameters. This is in contrast with the case where f and G are functions taking values in the space of (closed) subsets of \mathbb{R}^d which is discussed in [15, 18, 19, 20, 27, 28]. Note that the latter

approach could also be applied to the case of single-valued coefficients involving set-valued (even time dependent) parameters. But one substantial restriction is that a set-valued coefficient G in the noise term can lead to unbounded random sets in the solution process (even in very simple examples - see [27], Theorem 1) whereas using the method proposed in this paper leads to compact values when random compact sets are used to model parameter uncertainty. Of course, instead of random sets we could use fuzzy sets. But since each fuzzy set can be interpreted as a consonant random set on the interval $[0, 1]$ as underlying probability space, dealing with random sets is more general.

2 Preliminaries

2.1 Stochastic Processes

Throughout this section let (Ω, Σ, P) denote a probability space with σ -algebra Σ and probability measure P and let (T, r) and (\mathbb{E}, ρ) be metric spaces. A stochastic process is a map

$$x : T \times \Omega \rightarrow \mathbb{E}, \omega \mapsto x_t(\omega) = x(t, \omega)$$

such that for each $t \in T$ the map

$$x_t : \Omega \rightarrow \mathbb{E}, \omega \mapsto x_t(\omega)$$

is a random variable, that is, it is measurable. For fixed $\omega \in \Omega$ the map

$$x(\omega) : T \rightarrow \mathbb{E}, t \mapsto x_t(\omega)$$

is called sample function. Very often properties of stochastic processes cannot be verified for all $\omega \in \Omega$ but only for almost all ω , that is, for some subset of Ω whose probability is 1. That is why the term version is frequently used. Two stochastic processes x and \tilde{x} are called versions of each other (or stochastically equivalent) if for all $t \in T$ it holds that

$$P(\{\omega : x_t(\omega) = \tilde{x}_t(\omega)\}) = 1.$$

The first property that should be mentioned here is separability.

Definition 1. ([5, 11]) Suppose that (T, r) is separable. A stochastic process $x : T \times \Omega \rightarrow \mathbb{E}$ is said to be separable if there exists a dense countable subset D of T and a set $N \in \Sigma$ of measure 0 such that for each open subset $G \subseteq T$ and every closed subset $F \subseteq \mathbb{E}$ the two sets

$$\begin{aligned} \{\omega : \forall t \in G \cap D : x_t(\omega) \in F\} \\ \{\omega : \forall t \in G : x_t(\omega) \in F\} \end{aligned}$$

differ at most in N .

Hence, one could say that separability means that considering x for countably many $t \in T$ is enough to observe the behavior of the whole process. The following theorem whose proof can for example be found in [5] or [11] is fundamental for the theory of stochastic processes.

Theorem 1. ([5, 11]) Suppose that T is separable and \mathbb{E} is compact. Then for any stochastic process $x : T \times \Omega \rightarrow \mathbb{E}$ there is a separable version.

Note that if \mathbb{E} is only locally compact (which is the case if $\mathbb{E} = \mathbb{R}^d$) then one can always find a separable version in some compactification of \mathbb{E} and its values are still in \mathbb{E} with probability 1 for each $t \in T$.

Definition 2. A stochastic process is called (almost surely) continuous if (almost) all sample functions are continuous.

Recall that a probability space (Ω, Σ, P) is said to be complete if all subsets of sets $N \in \Sigma$ with $P(N) = 0$ are measurable, that is, lie in Σ . The completion of a probability space (Ω, Σ, P) is denoted $(\Omega, \overline{\Sigma}^P, \overline{P})$.

Proposition 1. ([10]) Suppose that (T, r) is separable and (Ω, Σ, P) is complete. Then a separable stochastic process which has an almost surely continuous version is almost surely continuous itself.

The next theorem states the so-called Kolmogorov-Chentsov criterion for almost sure continuity of sample functions.

Theorem 2. ([16]) Let $T = \mathbb{R}^p$, let (\mathbb{E}, ρ) be a complete metric space. Suppose that a process $x : T \times \Omega \rightarrow \mathbb{E}$ satisfies for some positive constants α, β, γ the following condition

$$\mathbb{E}(\rho(x_s, x_t)^\alpha) \leq \gamma \|s - t\|^{p+\beta} \quad \forall s, t \in T = \mathbb{R}^p. \quad (3)$$

Then x has an almost surely continuous version.

In the situation of the above Theorem 2, separability of x implies almost sure continuity of x if (Ω, Σ, P) is complete.

Definition 3. A stochastic process $x : T \times \Omega \rightarrow \mathbb{E}$ is called measurable if x is a measurable function with respect to the product- σ -algebra $\mathcal{B}(T) \otimes \Sigma$ where $\mathcal{B}(T)$ denotes the Borel- σ -algebra of (T, r) .

Theorem 3. ([13]) Suppose that T is separable. Then a continuous process $x : T \times \Omega \rightarrow \mathbb{E}$ is measurable.

In the case where it is only known that almost all sample functions are continuous one can construct a version possessing only continuous sample functions by choosing a continuous sample path and replacing all discontinuous sample functions with this path.

2.2 Random Sets

A random set is a random variable whose values are sets. It is usual to consider random closed sets, that is, random variables whose values are closed subsets of some topological space \mathbb{E} . The Borel- σ -algebra on \mathbb{E} is denoted by $\mathcal{B}(\mathbb{E})$ while $\mathcal{G}(\mathbb{E})$, $\mathcal{F}(\mathbb{E})$ and $\mathcal{K}(\mathbb{E})$ denote, respectively, the family of open, closed and compact subsets of \mathbb{E} . By $\mathcal{F}'(\mathbb{E})$ and $\mathcal{K}'(\mathbb{E})$ we mean $\mathcal{F}(\mathbb{E}) \setminus \{\emptyset\}$ and $\mathcal{K}(\mathbb{E}) \setminus \{\emptyset\}$, respectively.

Again let (Ω, Σ, P) be a probability space. As with random variables a random closed set $A : \Omega \rightarrow \mathcal{F}(\mathbb{E})$ has to fulfill some measurability condition. We shall demand that

$$A^-(B) = \{\omega : A(\omega) \cap B \neq \emptyset\} \in \Sigma, \quad \forall B \in \mathcal{B}(\mathbb{E}). \quad (4)$$

For other measurability definitions for set-valued maps we refer to [2, 13], for example. Furthermore, we call A a random compact set if Condition (4) is satisfied and for all $\omega \in \Omega$ it holds that $A(\omega) \in \mathcal{K}(\mathbb{E})$.

One can view a random set A as a collection of random variables that fit inside A . Such single-valued measurable functions $\alpha : \Omega \rightarrow \mathbb{E}$ fulfilling

$$\alpha(\omega) \in A(\omega), \quad \forall \omega \in \Omega$$

are called selections of A . Let $\mathcal{S}(A)$ denote the set of all measurable selections of A . The following theorem which is referred to as the Fundamental Measurability Theorem gives conditions for the measurability of random closed sets and the existence of measurable selections. For its proof and related results see [2] and [13].

Theorem 4. ([2, 13]) Suppose that (\mathbb{E}, ρ) is a complete separable metric space. Let $A : \Omega \rightarrow \mathcal{F}'(\mathbb{E})$ be a set-valued mapping with non-empty values. Consider the following properties:

- (i) For all $B \in \mathcal{B}(\mathbb{E})$ it holds that $A^-(B) \in \Sigma$,
- (ii) for all $F \in \mathcal{F}(\mathbb{E})$ it holds that $A^-(F) \in \Sigma$,
- (iii) for all $G \in \mathcal{G}(\mathbb{E})$ it holds that $A^-(G) \in \Sigma$,
- (iv) there is a Castaing representation of A , that is, a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of measurable selections such that for all $\omega \in \Omega$

$$A(\omega) = \text{cl}(\{\alpha_n(\omega)\}_{n \in \mathbb{N}})$$

where cl denotes the closure in \mathbb{E} ,

- (v) for all $x \in \mathbb{E}$ the function $\omega \mapsto \inf_{y \in A(\omega)} \rho(x, y)$ is measurable,

- (vi) the graph of A belongs to $\Sigma \otimes \mathcal{B}(\mathbb{E})$.

Then the following implications hold:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Rightarrow (vi)$$

If (Ω, Σ, P) is a complete probability space then all properties are equivalent.

Note that in the literature (for example in [22]) one can also find “almost all” versions of the above theorem and definitions. For further background information on random sets see [21, 22, 29].

3 Stochastic Differential Equations with Random Set Parameters

3.1 Deterministic parameters

Let us consider stochastic differential equations of the form (2) whose initial value x_{t_0} and coefficients f and G depend on some vector $a = (a_1, \dots, a_p) \in \mathbb{A}$ of parameters where $\mathbb{A} \subseteq \mathbb{R}^p$ denotes the set of possible parameter values, that is, we consider differential equations of the form

$$x_{t,a} = x_{t_0,a} + \int_{t_0}^t f(s, a, x_{s,a}) ds + \int_{t_0}^t G(s, a, x_{s,a}) dw_s \quad (5)$$

where $t_0 \leq t \leq \bar{t} < \infty$, $a \in \mathbb{A}$, w_t denotes an m -dimensional Wiener process on a probability space (Ω, Σ, P) and

$$\begin{aligned} x_{t_0} : \mathbb{A} \times \Omega &\rightarrow \mathbb{R}^d, & (a, \omega) &\mapsto x_{t_0,a}(\omega), \\ f : [t_0, \bar{t}] \times \mathbb{A} \times \mathbb{R}^d &\rightarrow \mathbb{R}^d, & (t, a, x) &\mapsto f(t, a, x), \\ G : [t_0, \bar{t}] \times \mathbb{A} \times \mathbb{R}^d &\rightarrow \mathbb{R}^{d \times m}, & (t, a, x) &\mapsto G(t, a, x). \end{aligned}$$

Assume that for each $a \in \mathbb{A}$ the partial maps $f(\cdot, a, \cdot)$ and $G(\cdot, a, \cdot)$ are measurable functions and the usual conditions for the existence of a solution process ([1, 12]) are fulfilled, that is,

- (IV) $x_{t_0,a}$ is a random variable independent of the increments $w_t - w_{t_0}$ for $t \geq t_0$.
- (Lip) Lipschitz condition: There is a constant $L > 0$ such that for all $t \in [t_0, \bar{t}]$ and all $x, y \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &\|f(t, a, x) - f(t, a, y)\| \\ &+ \|G(t, a, x) - G(t, a, y)\| \leq L\|x - y\|. \end{aligned}$$

- (RG) Restriction on growth: There is a constant $K > 0$ such that for all $t \in [t_0, \bar{t}]$ and all $x \in \mathbb{R}^d$ it holds that

$$\|f(t, a, x)\|^2 + \|G(t, a, x)\|^2 \leq K(1 + \|x\|^2).$$

Note that the constants L and K can depend on a . If the above conditions are fulfilled we get for each $a \in \mathbb{A}$ a solution process $\{x_t\}_{t \in [t_0, \bar{t}]} = \{x_{t,a}\}_{t \in [t_0, \bar{t}]}$, which leads to a map of the form

$$x : [t_0, \bar{t}] \times \mathbb{A} \times \Omega \rightarrow \mathbb{R}^d, (t, a, \omega) \mapsto x_{t,a}(\omega). \quad (6)$$

Since for each $a \in \mathbb{A}$ and each $t \in [t_0, \bar{t}]$ the partial map $x_{t,a} = x(t, a, \cdot) : \Omega \rightarrow \mathbb{R}^d$ is measurable, (6) can be interpreted as a stochastic process on $[t_0, \bar{t}] \times \mathbb{A}$ which is a metric space. Hence, according to Theorem 1, we can assume x to be separable.

Looking at the process x defined by Equation (6) the question arises if it is continuous in (t, a) . From Itô's theory it is well-known that for fixed $a \in \mathbb{A}$ the solution process $\{x_{t,a}\}_{t \in [t_0, \bar{t}]}$ is continuous in t . Furthermore, it fulfills the inequality in Theorem 2 (see [1] or [12]), that is, there is some constant C such that for all $s, t \in [t_0, \bar{t}]$

$$\mathbb{E}(\|x_t - x_s\|^{2n}) \leq C|t - s|^n, \quad t, s \in [t_0, \bar{t}] \quad (7)$$

holds if the $2n$ -th moment of the initial value is finite. The next proposition will give conditions under which the corresponding inequality with respect to t and a is fulfilled on a bounded subset of $[t_0, \bar{t}] \times \mathbb{A}$.

Proposition 2. Let $\{x_{t,a}\}_{(t,a) \in [t_0, \bar{t}] \times \mathbb{A}}$ denote the process defined by Equation (6), let $U \subseteq \mathbb{A}$ be an arbitrary bounded subset of \mathbb{A} and let $n \in \mathbb{N}$. Assume that Conditions (IV), (Lip) and (RG) are fulfilled and in addition, the following conditions hold:

(C1) $L : \mathbb{A} \rightarrow \mathbb{R}_{\geq 0}$ from (Lip) and $K : \mathbb{A} \rightarrow \mathbb{R}_{\geq 0}$ from (RG) are bounded on U .

(C2) Local Lipschitz condition with respect to a : For all $x \in \mathbb{R}^d$ there exists a constant $\tilde{L} = \tilde{L}(U, x) > 0$ such that for all $t \in [t_0, \bar{t}]$ and for all $a, b \in U$ it holds that

$$\begin{aligned} & \|f(t, a, x) - f(t, b, x)\| \\ & + \|G(t, a, x) - G(t, b, x)\| \leq \tilde{L}(U, x)\|a - b\| \end{aligned}$$

where the growth of \tilde{L} is bounded by a polynomial in $\|x\|$, that is, there is an $M = M(U) > 0$ and a $k = k(U) \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$

$$\tilde{L}(U, x) \leq M(U)(1 + \|x\|)^k.$$

(C3) The $2nk$ -th moments of the initial values $x_{t_0,a}$ exist and are bounded on U , that is,

$$\sup_{a \in U} \mathbb{E}(\|x_{t_0,a}\|^{2nk}) < \infty.$$

In addition, there is a constant $c = c(U, n)$ such that for all $a, b \in U$ it holds that

$$\mathbb{E}(\|x_{t_0,a} - x_{t_0,b}\|^{2n}) \leq c\|a - b\|^{2n}.$$

Then there is a constant $C = C(U, n) > 0$ such that for all $s, t \in [t_0, \bar{t}]$ and for all $a, b \in U$ the following inequality holds

$$\mathbb{E}(\|x_{s,a} - x_{t,b}\|^{2n}) \leq C \left\| \begin{pmatrix} s - t \\ a - b \end{pmatrix} \right\|^n. \quad (8)$$

The rather technical proof is omitted since it is similar to the proof of (7) (see [1, 12]).

Now, we can conclude that a separable version of our process (6) is almost surely continuous with respect to (t, a) if the conditions of the above proposition are satisfied for $n \in \mathbb{N}$ big enough.

Proposition 3. The stochastic process $\{x_{t,a}\}_{(t,a) \in [t_0, \bar{t}] \times \mathbb{A}}$ defined by (6) is almost surely continuous with respect to (t, a) if there is an $n \geq p + 2$ such that the conditions of Proposition 2 are satisfied for each bounded subset $U \subseteq \mathbb{A}$.

Proof. Let $c \in \mathbb{A}$ and let $U(c) \subseteq \mathbb{A}$ denote a bounded neighborhood of c . Since the conditions of Proposition 2 are fulfilled for some $n \geq p + 2$ we know that (8) holds for all $(s, a), (t, b) \in [t_0, \bar{t}] \times U(c)$ which means that, according to Proposition 1, x is an almost surely continuous process on $[t_0, \bar{t}] \times U(c)$, that is, there is a measure-zero set $N(c) \in \Sigma$ such that for all $\omega \in N(c)^c$ the sample function $x_{\cdot, \cdot}(\omega)$ is continuous. Since \mathbb{A} can be covered by bounded neighborhoods of countably many $c \in \mathbb{A}$ the set $N^c = \bigcap_c N(c)^c$ is measurable and has probability 1 which means that x is an almost surely continuous process on $[t_0, \bar{t}] \times \mathbb{A}$. \square

If we replace, as described at the end of Section 2.1, $\{x_{t,a}\}_{(t,a) \in [t_0, \bar{t}] \times \mathbb{A}}$ by a continuous version, we can infer measurability from Theorem 3.

Corollary 1. Let $\mathbb{A} \in \mathcal{B}(\mathbb{R}^p)$ be a Borel set and let $\{x_{t,a}\}_{(t,a) \in [t_0, \bar{t}] \times \mathbb{A}}$ be a continuous process of the form (6). If we choose an $\omega \in \Omega$ such that $x_{\cdot, \cdot}(\omega)$ is continuous and replace all discontinuous sample functions by $x_{\cdot, \cdot}(\omega)$ we get a continuous version which is, according to Theorem 3, measurable with respect to $\mathcal{B}([t_0, \bar{t}]) \otimes \mathcal{B}(\mathbb{A}) \otimes \Sigma$.

3.2 Parameters modelled by random variables

From now on the probability space on which the Wiener process $\{w_t\}_{t \geq t_0}$ is defined shall be denoted $(\Omega_w, \Sigma_w, P_w)$. We assume that the stochastic process $\{x_{t,a}\}_{(t,a) \in [t_0, \bar{t}] \times \mathbb{A}}$ defined by Equation (6) is measurable with respect to the product- σ -algebra $\mathcal{B}([t_0, \bar{t}]) \otimes \mathcal{B}(\mathbb{A}) \otimes \Sigma_w$ and all sample functions are continuous on $[t_0, \bar{t}] \times \mathbb{A}$. The measurability of x allows us to model the parameter uncertainty of a by a random

variable, that is, a measurable function $\alpha : \Omega_{\mathbb{A}} \rightarrow \mathbb{A}$ on some probability space $(\Omega_{\mathbb{A}}, \Sigma_{\mathbb{A}}, P_{\mathbb{A}})$. Consequently, the map

$$\hat{\alpha} : \begin{array}{l} [t_0, \bar{t}] \times \Omega_{\mathbb{A}} \times \Omega_w \\ (t, \omega_{\mathbb{A}}, \omega_w) \end{array} \mapsto \begin{array}{l} [t_0, \bar{t}] \times \mathbb{A} \times \Omega_w \\ (t, \alpha(\omega_{\mathbb{A}}), \omega_w) \end{array}$$

is measurable with respect to the product σ -algebra $\mathcal{B}([t_0, \bar{t}]) \otimes \Sigma_{\mathbb{A}} \otimes \Sigma_w$. Composing $\hat{\alpha}$ and x leads to the measurable map $\xi = x \circ \hat{\alpha}$

$$\xi : \begin{array}{l} [t_0, \bar{t}] \times \Omega_{\mathbb{A}} \times \Omega_w \\ (t, \omega_{\mathbb{A}}, \omega_w) \end{array} \mapsto \begin{array}{l} \mathbb{R}^d \\ x(t, \alpha(\omega_{\mathbb{A}}), \omega_w) \end{array} \quad (9)$$

which can be interpreted as a stochastic process $\{\xi_t\}_{t \in [t_0, \bar{t}]}$ on the time interval $[t_0, \bar{t}]$ and the product space $(\Omega, \Sigma, P) = (\Omega_{\mathbb{A}} \times \Omega_w, \Sigma_{\mathbb{A}} \otimes \Sigma_w, P_{\mathbb{A}} \otimes P_w)$.

Proposition 4. The map ξ defined by (9) can be interpreted as a stochastic process $\{\xi_t\}_{t \in [t_0, \bar{t}]}$ on the time interval $[t_0, \bar{t}]$ and the probability space (Ω, Σ, P) . The process $\{\xi_t\}_{t \in [t_0, \bar{t}]}$ is measurable and all sample functions are continuous.

Proof. The map $\xi = x \circ \hat{\alpha}$ is measurable since it is the composition of the two measurable functions $\hat{\alpha}$ and x where the domain of x is the same measure space as the range of $\hat{\alpha}$. Consequently, for each $t \in [t_0, \bar{t}]$ the partial map

$$\xi_t : \Omega \rightarrow \mathbb{R}^d, \omega \mapsto x_{t, \alpha(\omega_{\mathbb{A}})}(\omega_w)$$

is a random variable which means that ξ is a measurable stochastic process. Note that for each $a \in \mathbb{A}$ and each $\omega_w \in \Omega_w$ the partial map $x_{\cdot, a}(\omega_w)$ is continuous because the sample function $x_{\cdot, \cdot}(\omega_w)$ is continuous. Since for all $\omega_{\mathbb{A}}$ we have $\alpha(\omega_{\mathbb{A}}) \in \mathbb{A}$ we can infer that $\xi_t(\omega) = x_{\cdot, \alpha(\omega_{\mathbb{A}})}(\omega_w)$ is continuous for all $\omega \in \Omega$. \square

3.3 Parameters modelled by random sets

The uncertainty of the parameter a in Equation (5) shall now be modelled by a random compact set

$$A : \Omega_{\mathbb{A}} \rightarrow \mathcal{K}'(\mathbb{A})$$

where $\mathcal{K}'(\mathbb{A})$ denotes the set of all non-empty compact subsets of \mathbb{R}^p being also a subset of \mathbb{A} . Then we can define a set-valued function X by

$$X : (t, \omega) \mapsto \{x_{t, a}(\omega_w) : a \in A(\omega_{\mathbb{A}})\} \quad (10)$$

where $(t, \omega) \in [t_0, \bar{t}] \times \Omega$ and x is the process defined by (6) which is still assumed to be measurable and continuous. The next proposition states that X is a set-valued process with compact values, that is, for each $t \in [t_0, \bar{t}]$ it holds that X_t is a random compact set which particularly means that the measurability condition (4) is fulfilled.

Proposition 5. Let $A : \Omega_{\mathbb{A}} \rightarrow \mathcal{K}'(\mathbb{A})$ be a random compact set and let X be the set-valued map defined by Equation (10). Then the following holds:

1. X can be interpreted as a set-valued process on the time interval $[t_0, \bar{t}]$ and the completed probability space $(\Omega, \overline{\Sigma}^P, \overline{P})$ with values in $\mathcal{K}'(\mathbb{R}^d)$,
2. All sample functions of X are continuous with respect to the Hausdorff-metric H on $\mathcal{K}'(\mathbb{R}^d)$.
3. X is measurable with respect to the product- σ -algebra $\mathcal{B}([t_0, \bar{t}]) \otimes \overline{\Sigma_{\mathbb{A}}} \otimes \overline{\Sigma_w}^{P_{\mathbb{A}} \otimes P_w}$.
4. For a Castaing representation $\{\alpha_n\}_{n \in \mathbb{N}}$ of A the processes $\{\xi^n\}_{n \in \mathbb{N}}$ defined by

$$\xi_t^n(\omega) = x_{t, \alpha_n(\omega_{\mathbb{A}})}(\omega_w), (t, \omega) \in [t_0, \bar{t}] \times \Omega$$

form a Castaing representation of X and for each $t \in [t_0, \bar{t}]$ the family $\{\xi_t^n\}_{n \in \mathbb{N}}$ forms a Castaing representation of X_t .

Proof. First note that $X_t(\omega)$ is a non-empty compact subset of \mathbb{R}^d for all $t \in [t_0, \bar{t}]$ and all $\omega \in \Omega$ since $x_{t, \cdot}(\omega_w)$ is continuous in a and $A(\omega_{\mathbb{A}})$ is a non-empty compact subset of \mathbb{R}^p for all $\omega_{\mathbb{A}} \in \Omega_{\mathbb{A}}$. Since for the proof of the first three statements the Castaing representation $\{\xi^n\}_{n \in \mathbb{N}}$ is used Assertion 4 is proved first. Hence, we show that for all $(t, \omega) \in [t_0, \bar{t}] \times \Omega$ it holds that

$$\{x_{t, a}(\omega_w) : a \in A(\omega_{\mathbb{A}})\} = \text{cl}(\{\xi_t^n(\omega)\}_{n \in \mathbb{N}}).$$

In fact, since $\{\alpha_n\}_{n \in \mathbb{N}}$ is a Castaing representation of A we know that for all $a \in A(\omega_{\mathbb{A}})$ there is a subsequence $\{\alpha_{n_j}\}_{j \in \mathbb{N}}$ such that $\alpha_{n_j}(\omega_{\mathbb{A}}) \rightarrow a$ for $j \rightarrow \infty$. Continuity of $x_{t, \cdot}(\omega_w)$ in a implies $\xi_t^{n_j}(\omega) = x_{t, \alpha_{n_j}(\omega_{\mathbb{A}})}(\omega_w) \rightarrow x_{t, a}(\omega_w)$ which means that $x_{t, a}(\omega_w) \in \text{cl}(\{\xi_t^n(\omega)\})$. On the other hand, it is clear that $\alpha_n(\omega_{\mathbb{A}}) \in A(\omega_{\mathbb{A}})$ for all $\omega_{\mathbb{A}} \in \Omega_{\mathbb{A}}$, $n \in \mathbb{N}$ and consequently $\xi_t^n(\omega) \in X_t(\omega)$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. Since $X_t(\omega)$ is closed, it holds that $\text{cl}(\{\xi_t^n(\omega)\}_{n \in \mathbb{N}}) \subseteq X_t(\omega)$. Hence, for each $t \in [t_0, \bar{t}]$ it follows that $X_t(\omega) = \text{cl}(\{\xi_t^n(\omega)\}_{n \in \mathbb{N}})$ for all $\omega \in \Omega$. According to the Fundamental Measurability Theorem 4, this means that X_t is a random compact set on the completion of the probability space (Ω, Σ, P) , that is,

$$(\Omega_{\mathbb{A}} \times \Omega_w, \overline{\Sigma_{\mathbb{A}}} \otimes \overline{\Sigma_w}^{P_{\mathbb{A}} \otimes P_w}, \overline{P_{\mathbb{A}} \otimes P_w}).$$

The continuity of X is a consequence of the continuity of the processes ξ^n ($n \in \mathbb{N}$). Indeed, after recalling that for $A, B \in \mathcal{K}'(\mathbb{R}^d)$ the Hausdorff-metric H is defined by

$$H(A, B) = \max(\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|)$$

suppose that for arbitrary $\omega \in \Omega$ there is a $t \in [t_0, \bar{t}]$ and an $\varepsilon_0 > 0$ such that for all $\delta > 0$ there is an $s = s(\delta)$ such that $|s - t| < \delta$ and

$$H(X_s(\omega), X_t(\omega)) \geq \varepsilon_0.$$

Because of the closedness of $X_t(\omega)$ and $X_s(\omega)$ this corresponds to the assumption that at least one of the following two inequalities holds

$$\begin{aligned} \sup_{n \in \mathbb{N}} \inf_{m \in \mathbb{N}} \|\xi_s^n(\omega) - \xi_t^m(\omega)\| &\geq \varepsilon_0, \\ \sup_{m \in \mathbb{N}} \inf_{n \in \mathbb{N}} \|\xi_s^n(\omega) - \xi_t^m(\omega)\| &\geq \varepsilon_0. \end{aligned}$$

From the first inequality one can infer that there is an $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ it holds that

$$\|\xi_s^n(\omega) - \xi_t^m(\omega)\| \geq \inf_{m \in \mathbb{N}} \|\xi_s^n(\omega) - \xi_t^m(\omega)\| \geq \frac{\varepsilon_0}{2}.$$

Of course, this inequality also holds for the choice $m = n$ which leads to

$$\|\xi_s^n(\omega) - \xi_t^n(\omega)\| \geq \frac{\varepsilon_0}{2},$$

but this would mean that ξ^n is not continuous at t . If we apply the same argument to the second inequality we can conclude that $H(X_s(\omega), X_t(\omega)) \geq \varepsilon_0$ cannot hold. Hence, X is a continuous process.

Since $\mathcal{K}'(\mathbb{R}^d)$ together with the Hausdorff metric H is a metric space the measurability of X is a direct consequence of the continuity of all sample functions $X_t(\omega)$ and Theorem 3. \square

The different maps that appeared in this section together with the underlying measure spaces are summarized in the following table. (Note that λ and λ^p denote the Lebesgue measures on $\mathcal{B}([t_0, \bar{t}])$ and $\mathcal{B}(\mathbb{A})$, respectively.)

map	underlying measure space
x	$([t_0, \bar{t}] \times \mathbb{A} \times \Omega_w, \mathcal{B}([t_0, \bar{t}]) \otimes \mathcal{B}(\mathbb{A}) \otimes \Sigma_w, \lambda \otimes \lambda^p \otimes P_w)$
α, A	$(\Omega_{\mathbb{A}}, \Sigma_{\mathbb{A}}, P_{\mathbb{A}})$
$\hat{\alpha}, \xi$	$([t_0, \bar{t}] \times \Omega_{\mathbb{A}} \times \Omega_w, \mathcal{B}([t_0, \bar{t}]) \otimes \Sigma_{\mathbb{A}} \otimes \Sigma_w, \lambda \otimes P_{\mathbb{A}} \otimes P_w)$
X	$([t_0, \bar{t}] \times \Omega_{\mathbb{A}} \times \Omega_w, \mathcal{B}([t_0, \bar{t}]) \otimes \overline{\Sigma_{\mathbb{A}} \otimes \Sigma_w}^{P_{\mathbb{A}} \otimes P_w}, \lambda \otimes \overline{P_{\mathbb{A}} \otimes P_w})$

4 First Entrance and Inclusion Times for Set-valued Processes

In many applications, it is useful to observe the first time where a single-valued stochastic process enters

some subset of the state space or the last time where it leaves this subset. For example, one could be interested in the first exceedance of a certain level by a real-valued process to assess the reliability of a system described by this process (see for example [30]). In his book [7], Dynkin discusses the theory of first entrance and exit times of right-continuous Markov processes. Other theoretical background can be found in [4, 17].

For a (single-valued) d -dimensional process $\{\xi_t\}_{t \in [t_0, \bar{t}]}$ on a probability space (Ω, Σ, P) and a subset $B \subseteq \mathbb{R}^d$ we shall call

$$\tau_{\xi}^B : \Omega \rightarrow [t_0, \bar{t}], \omega \mapsto \inf\{t : \xi_t(\omega) \in B\} \quad (11)$$

the first entrance time of ξ into B . Note that if the infimum does not exist we set $\tau_{\xi}^B(\omega) = \bar{t}$. One can show (see [7]) that (11) is measurable if $B \in \mathcal{B}(\mathbb{R}^d)$ and ξ is right-continuous. Furthermore, if ξ is continuous and B is a closed subset of \mathbb{R}^d then τ_{ξ}^B is a stopping time w.r.t. the natural filtration $\{\mathcal{A}_t\}_{t \in [t_0, \bar{t}]}$ defined by

$$\mathcal{A}_t = \sigma(\xi_s^{-1}(B) : s \in [t_0, \bar{t}], B \in \mathcal{B}(\mathbb{R}^d)), \quad (12)$$

and if B is open then τ_{ξ}^B is a stopping time w.r.t. the right-continuous filtration $\{\mathcal{A}_{t+}\}_{t \in [t_0, \bar{t}]}$ where

$$\mathcal{A}_{t+} = \bigcap_{t < s \leq \bar{t}} \mathcal{A}_s, \mathcal{A}_{\bar{t}+} = \mathcal{A}_{\bar{t}}. \quad (13)$$

If we consider a continuous process $\{X_t\}_{t \in [t_0, \bar{t}]}$ with values in $\mathcal{K}'(\mathbb{R}^d)$ we can define the following two maps that correspond to (11):

$$\underline{\tau}^B : \Omega \rightarrow [t_0, \bar{t}], \omega \mapsto \inf\{t : X_t(\omega) \cap B \neq \emptyset\} \quad (14)$$

$$\overline{\tau}^B : \Omega \rightarrow [t_0, \bar{t}], \omega \mapsto \inf\{t : X_t(\omega) \subseteq B\} \quad (15)$$

If the infimum does not exist, we set $\underline{\tau}^B(\omega) = \bar{t}$ or $\overline{\tau}^B(\omega) = \bar{t}$, respectively. We call $\underline{\tau}^B$ the first entrance time of X into B , and we call $\overline{\tau}^B$ the first inclusion time of X in B .

Considering the natural filtration $\{\Sigma_t\}_{t \in [t_0, \bar{t}]}$ of X defined by

$$\Sigma_t = \sigma(X_s^-(B) : s \in [t_0, t], B \in \mathcal{B}(\mathbb{R}^d)) \subseteq \Sigma \quad (16)$$

the next proposition (which is the set-valued analogue of Dynkin's Lemma 4.1 in [7]) gives conditions under which $\underline{\tau}^B$ and $\overline{\tau}^B$ are measurable or even stopping times w.r.t. the augmented filtration $\{\hat{\Sigma}_t^P\}_{t \in [t_0, \bar{t}]}$, that is the ascending family of complete σ -algebras defined by

$$\hat{\Sigma}_t^P = \sigma(\Sigma_t \cup \mathcal{N}) \subseteq \overline{\Sigma}^P \quad (17)$$

where \mathcal{N} is the set of all subsets of measure-zero sets in Σ .

Proposition 6. Suppose that $\{X_t\}_{t \in [t_0, \bar{t}]}$ is a continuous $\mathcal{K}'(\mathbb{R}^d)$ -valued process on a probability space (Ω, Σ, P) and $\{\Sigma_t\}_{t \in [t_0, \bar{t}]}$ is its natural filtration defined by (16).

1. If $B \in \mathcal{G}(\mathbb{R}^d)$ is an open subset of \mathbb{R}^d then

$$\{\omega : \underline{\tau}^B(\omega) \leq t\}, \{\omega : \bar{\tau}^B(\omega) \leq t\} \in \hat{\Sigma}_{t+}^P.$$

2. If $B \in \mathcal{F}(\mathbb{R}^d)$ is a closed subset of \mathbb{R}^d then

$$\{\omega : \underline{\tau}^B(\omega) \leq t\}, \{\omega : \bar{\tau}^B(\omega) \leq t\} \in \hat{\Sigma}_t^P.$$

Proof. The proof is omitted here since it is very similar to the proof of Lemma 4.1 in [7]. \square

An interesting question is if $\underline{\tau}^B$ and $\bar{\tau}^B$ can be attained by first entrance times of selections of X . The next proposition states that this is possible.

Proposition 7. Let $X : [t_0, \bar{t}] \times \Omega \rightarrow \mathcal{K}'(\mathbb{R}^d)$ be a continuous set-valued process with non-empty compact values and let $B \subseteq \mathbb{R}^d$ be an arbitrary subset of \mathbb{R}^d . Then for all $\omega \in \Omega$ it holds that

$$\begin{aligned} \inf_{\xi \in \mathcal{S}(X)} \tau_{\xi}^B(\omega) &= \underline{\tau}^B(\omega), \\ \sup_{\xi \in \mathcal{S}(X)} \tau_{\xi}^B(\omega) &\leq \bar{\tau}^B(\omega). \end{aligned}$$

If (Ω, Σ, P) is complete and $B \in \mathcal{G}(\mathbb{R}^d)$ then for all $\omega \in \Omega$ the second inequality becomes an equality.

Proof. The equality for $\underline{\tau}^B$ and the inequality for $\bar{\tau}^B$ can be seen easily by using the equation

$$X_t(\omega) = \{\xi_t(\omega) : \xi \in \mathcal{S}(X)\}$$

which holds for all $t \in [t_0, \bar{t}]$ and $\omega \in \Omega$. If (Ω, Σ, P) is complete and B is an open subset of \mathbb{R}^d then $\bar{\tau}^B$ is Σ -measurable by Proposition 6. Consider the map

$$Y : (t, \omega) \mapsto \begin{cases} X_t(\omega) & \text{if } \bar{\tau}^B(\omega) \leq t \\ X_t(\omega) \cap B^c & \text{if } \bar{\tau}^B(\omega) > t \end{cases}$$

which has non-empty closed values. Note that

$$M = \{(t, \omega) \in [t_0, \bar{t}] \times \Omega : \bar{\tau}^B(\omega) \leq t\} \in \mathcal{B}([t_0, \bar{t}]) \otimes \Sigma$$

since $(t, \omega) \mapsto \bar{\tau}^B(\omega) - t$ is a measurable function. Furthermore, it can be checked easily that for any $C \in \mathcal{B}(\mathbb{R}^d)$ it holds that

$$Y^-(C) = (X^-(C) \cap M) \cup (X^-(B^c \cap C) \cap M^c)$$

which means that Y is a random closed set. From Theorem 4 one can infer that there is a selection $\xi \in \mathcal{S}(Y)$ which implies that $\tau_{\xi}^B(\omega) = \bar{\tau}^B(\omega)$ for all $\omega \in \Omega$. Since $Y(\omega) \subseteq X(\omega)$ for all $\omega \in \Omega$ the map ξ is also a selection of X . \square

For a set-valued process defined by (10) which fulfills the conditions of Proposition 5 we can consider for each $\alpha \in \mathcal{S}(A)$ and $a \in \mathbb{A}$ the special entrance times

$$\begin{aligned} \tau_{\alpha}^B : \omega &\mapsto \inf\{t \in [t_0, \bar{t}] : x_{t, \alpha(\omega_{\mathbb{A}})}(\omega_w) \in B\}, \\ \tau_a^B : \omega_w &\mapsto \inf\{t \in [t_0, \bar{t}] : x_{t, a}(\omega_w) \in B\}. \end{aligned}$$

Proposition 8. Let $X : [t_0, \bar{t}] \times \Omega \rightarrow \mathcal{K}'(\mathbb{R}^d)$ be a set-valued process defined by (10) which fulfills the conditions of Proposition 5. Then the following relations hold for all $\omega \in \Omega$

$$\begin{aligned} \inf_{a \in A(\omega_{\mathbb{A}})} \tau_a^B(\omega_w) &= \inf_{\alpha \in \mathcal{S}(A)} \tau_{\alpha}^B(\omega) = \inf_{\xi \in \mathcal{S}(X)} \tau_{\xi}^B(\omega) \\ \sup_{a \in A(\omega_{\mathbb{A}})} \tau_a^B(\omega_w) &= \sup_{\alpha \in \mathcal{S}(A)} \tau_{\alpha}^B(\omega) \leq \sup_{\xi \in \mathcal{S}(X)} \tau_{\xi}^B(\omega). \end{aligned}$$

Proof. Let $\omega \in \Omega$. Note that $\tau_{\alpha}^B(\omega) = \tau_{\alpha(\omega_{\mathbb{A}})}^B(\omega_w)$ for all $\alpha \in \mathcal{S}(A)$ and $A(\omega_{\mathbb{A}}) = \{\alpha(\omega_{\mathbb{A}}) : \alpha \in \mathcal{S}(A)\}$. Then in both lines the left equality is obvious. According to Proposition 7 the second equality in the first line is proved by showing

$$\inf_{a \in A(\omega_{\mathbb{A}})} \tau_a^B(\omega_w) = \underline{\tau}^B(\omega).$$

From the relations $\{x_{\cdot, \alpha} : \alpha \in \mathcal{S}(A)\} \subseteq \mathcal{S}(X)$ and $\tau_{x_{\cdot, \alpha}}^B(\omega) = \tau_{\alpha}^B(\omega)$ we get the inequality in the second line. \square

This means that for processes of the form (10) the first entrance time $\underline{\tau}^B$ can be attained by observing the first entrance times of the special selections $x_{\cdot, a}$ or $x_{\cdot, \alpha}$. This can be useful for the practical calculation of $\underline{\tau}^B$. Unfortunately, there does not seem to be an obvious condition under which the attainability of $\bar{\tau}^B$ holds.

5 Example

In the following we shall give an illuminating example how the concept described in the foregoing sections can be applied to problems from structural mechanics where systems of ODEs of order one and two play an important role.

For the sake of simplicity we consider the so-called Langevin equation

$$dx_t = -a_1 x_t dt + a_2 dw_t$$

with initial value x_0 where w_t is a one dimensional Wiener process, $a_1 > 0$ and $a_2 \in \mathbb{R}$ ($d = m = 1$, $t_0 = 0$). Its unique solution is the so-called Ornstein-Uhlenbeck process

$$x_t = e^{-a_1 t} x_0 + a_2 \int_0^t e^{-a_1(t-s)} dw_s \quad (18)$$

which is a Gaussian stochastic process if and only if x_0 is normally distributed or constant. For modelling the uncertainty of the parameters a_1 and a_2 we shall use the following two finite random sets

$$\begin{aligned} A_1 : \quad & \omega_{A11} \mapsto [1, 3], & P_{A1}(\omega_{A11}) &= 2/5 \\ & \omega_{A12} \mapsto [2, 4], & P_{A1}(\omega_{A12}) &= 3/5 \\ A_2 : \quad & \omega_{A21} \mapsto [0.5, 1.5], & P_{A2}(\omega_{A21}) &= 1/3 \\ & \omega_{A22} \mapsto [1, 2], & P_{A2}(\omega_{A22}) &= 2/3 \end{aligned}$$

which can be written in the shorter form

$$\begin{aligned} A_1 &= \{([1, 3], 2/5), ([2, 4], 3/5)\}, \\ A_2 &= \{([0.5, 1.5], 1/3), ([1, 2], 2/3)\}. \end{aligned}$$

From these random sets we construct the following joint random set on a probability space $\Omega_A = \{\omega_{Ai}\}_{1 \leq i \leq 4}$ with values in $\mathcal{K}'(\mathbb{R}^2)$

$$A = \{([1, 3] \times [0.5, 1.5], 2/15), ([1, 3] \times [1, 2], 4/15), ([2, 4] \times [0.5, 1.5], 1/5), ([2, 4] \times [1, 2], 2/5)\}$$

by taking as focal elements the Cartesian products of each focal element of the first with each focal element of the second random set and multiplying the respective weights. This is a kind of independence which is called random set independence (see [3, 8, 9]). According to Equation (10) we get a set-valued process X with values in $\mathcal{K}'(\mathbb{R})$ which can be bounded by the single-valued processes L and U defined by

$$L_t(\omega) = \inf_{x \in X_t(\omega)} x, \quad U_t(\omega) = \sup_{x \in X_t(\omega)} x.$$

Furthermore, we consider the selection

$$\begin{aligned} \alpha : \quad & \omega_{A1} \mapsto (1.7, 1.1), & P_A(\omega_{A1}) &= 2/15 \\ & \omega_{A2} \mapsto (2.3, 1.5), & P_A(\omega_{A2}) &= 4/15 \\ & \omega_{A3} \mapsto (3.0, 0.9), & P_A(\omega_{A3}) &= 1/5 \\ & \omega_{A4} \mapsto (3.2, 1.4), & P_A(\omega_{A4}) &= 2/5 \end{aligned}$$

and the corresponding process ξ defined by (9).

Figure 1 shows details of sample functions of the boundary processes L and U (solid lines) with respect to the same sample function of the Wiener process and the choice $\omega_A = \omega_{A1}$. The dashed line shows the corresponding sample function of ξ . The graphs were simulated by using the Euler method (see for example [14]) with 1000 time steps from $t_0 = 0$ to $\bar{t} = 10$, $x_0 \equiv 0$. The interval $[1, 3] \times [0.5, 1.5]$ was discretized by a grid of 101×101 points applying to each of the grid points the Euler scheme and choosing in each time step the greatest value for U and the smallest value for L .

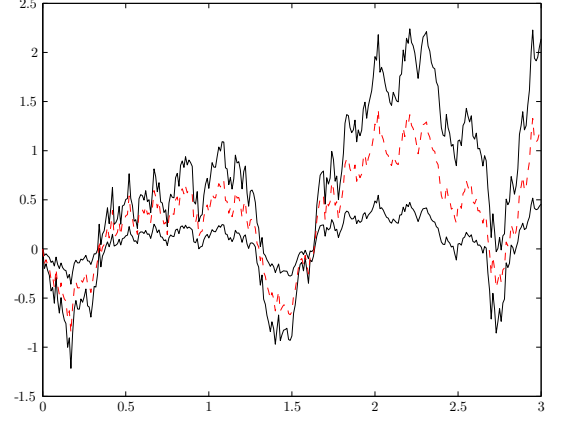


Figure 1: Sample functions of X (boundaries in solid lines) and ξ (dashed line).

Figure 2 shows upper and lower cumulative distribution functions

$$\begin{aligned} \underline{F}_t(x) &= P(X_t \subseteq (-\infty, x)) = P(U_t < x) \\ \overline{F}_t(x) &= P(X_t \cap (-\infty, x) \neq \emptyset) = P(L_t < x) \end{aligned}$$

of the random set X_t at time $t = 10$. They were calculated by simulating 1000 sample functions of the Wiener process and considering all four possible focal elements of A . The dashed line shows the cumulative distribution function $F_{t,\alpha}$ of the random variable ξ_t ($t = 10$).

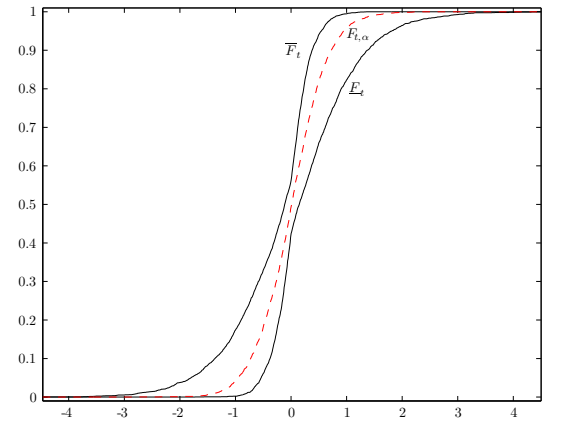


Figure 2: P-box of X_t (solid lines) and cumulative distribution function of ξ_t (dashed line) ($t = 10$).

Finally, one can consider the first entrance times $\underline{\tau}_\alpha^B$, τ_α^B and the first inclusion time $\bar{\tau}^B$ for $B = (0.5, \infty)$. The corresponding cumulative distribution functions are displayed in Figure 3.

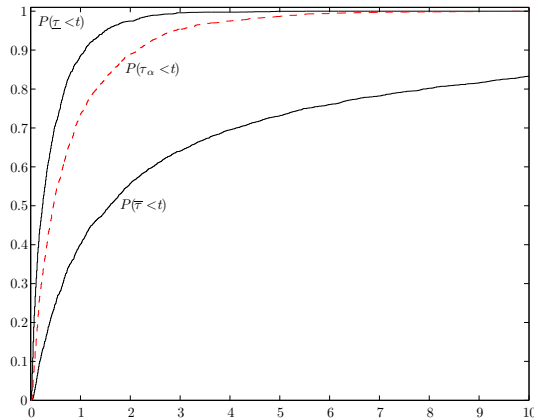


Figure 3: CDFs of first entrance time $\underline{\tau}^B$ and first inclusion time $\bar{\tau}^B$ (solid lines), CDF of first entrance time τ_α^B (dashed line).

6 Summary and Conclusions

In this paper, we consider ordinary stochastic differential equations whose coefficients depend on parameters. Conditions are given under which solution processes continuously depend on these parameters. If this is the case then modelling parameter uncertainty by using random compact sets leads to set-valued processes with compact values which are continuous with respect to the Hausdorff metric. We show that the single-valued solutions of the stochastic differential equation under scrutiny obtained by choosing single parameter values are selections which can be used to represent the set-valued process. Furthermore, analogues of first entrance times for set-valued processes are defined and their attainability by selections is discussed. Finally, an example is given to illustrate the theoretical concept.

As a topic for future research, we plan the investigation of further properties of the set-valued processes of the form (10). Furthermore, this theoretical concept will be applied to engineering problems (from structural mechanics) and it will be explored how first entrance and inclusion times (defined by (14), (15)) can be calculated or simulated.

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