Almost Bayesian Assignments and Conditional Independence (a contribution to Dempster-Shafer theory of evidence)

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Abstract

In the paper we introduce a family of almost Bayesian basic assignments, which slightly extends Bayesian basic assignments. This extension incorporates all the distributions that can be created from low-dimensional Bayesian basic assignments by application of the operator of composition, and simultaneously preserves the property of Bayesian basic assignments concerning the number of focal elements: it does not exceed cardinality of the frame of discernment. The other goal of the paper is to propagate a new way of definition of conditional independence relation in D-S theory. It follows ideas of P. P. Shenoy from [7], where the author defines the notion of conditional independence for valuation-based system based on his operation of "combination". Here we do the same but using the operator of "composition". The notion of independence we get in this way seems to meet better the general requirements on conditional independence relation for basic assignments.

Keywords. Dempster-Shafer theory of evidence, multidimensionality, operator of composition, conditional independence, semigraphoids.

1 Introduction

Regarding purely computational point of view, the greatest disadvantage of Dempster-Shafer theory of evidence (D-S) is that in contrast to probabilistic or possibilistic models, which can be described by the respective density functions (i.e. point functions), D-S models must be described by set functions. It means that while the number of necessary parameters of probabilistic or possibilistic models grows exponentially with the number of dimensions, for D-S models one needs a superexponential number of parameters.

It is known from theory of Bayesian networks (or graphical Markov models, in general) that the number of parameters can be drastically decreased by utilization of properties of conditional independence relations valid for the modelled situation. This was among the reasons why we designed an alternative approach for multidimensional probability distribution representation based on so called *operator of composition* [2]. The basic idea of these models is very simple: multidimensional models are assembled (composed) from a system of low-dimensional distributions by the operator of composition (in a specified order). Later on, Vejnarová introduced an analogous operator also for composition of possibility distributions and showed it manifested similar properties as its probabilistic counterpart [10, 11]. Recently we designed the operator of composition also for basic assignments in D-S theory of evidence [5] and proved that it met all the required properties necessary for multidimensional models representation [3, 4].

However, it is not the goal of this paper to publicize advantageous properties of the operator of composition for basic assignments. The goal of this contribution is twofold. The first one is to show that there exists a family of basic assignments, for specification of which one does not need more parameters than for probabilistic models and yet it enables modelling some type of ignorance (Section 4). The other goal is to show that if the conditional independence for basic assignments is defined with the help of the operator of composition (which was already done in [3]) one can prove semigraphoid axioms from a small number of operator's basic properties. This is done in Section 5.

2 Basic notion

Set notation

In the whole paper we shall deal with a finite number of variables $X_1, X_2, \ldots, X_n$ each of which is specified by a finite set $\mathbf{X}_i$ of its values. So, we will consider multidimensional space of discernment

$$\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n,$$

and its *subspaces*. For $K \subset N = \{1, 2, \ldots, n\}$, $\mathbf{X}_K$ denotes a Cartesian product of those $\mathbf{X}_i$, for which $i \in K$:

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i.$$

A *projection* of $x = (x_1, x_2, \ldots, x_n) \in \mathbf{X}_N$ into $\mathbf{X}_K$ will be denoted $x^{\downarrow K}$, i.e. for $K = \{i_1, i_2, \ldots, i_\ell\}$

$$x^{\downarrow K} = (x_{i_1}, x_{i_2}, \ldots, x_{i_\ell}) \in \mathbf{X}_K.$$

Analogously, for $K \subset L \subseteq N$ and $A \subset \mathbf{X}_L$, $A^{\downarrow K}$ will denote a *projection* of A into $\mathbf{X}_K$:

$$A^{\downarrow K} = \{y \in \mathbf{X}_K : \exists x \in A \ \ (y = x^{\downarrow K})\}.$$

Let us remark that we do not exclude situations when $K = \emptyset$. In this case $A^{\downarrow \emptyset} = \emptyset$.

Set $A \subseteq \mathbf{X}_K$ is said to be a *point-cylinder* if it can be expressed as a Cartesian product of a singleton and a subspace $\mathbf{X}_L$. More precisely: a point-cylinder is a set $A \subseteq \mathbf{X}_K$ for which there exists an index set (possibly empty) $L \subseteq K$ such that $|C^{\downarrow L}| \leq 1$ and

$$C = C^{\downarrow L} \times \mathbf{X}_{K \setminus L}.$$

Let us stress that if $L = \emptyset$ then $C = \mathbf{X}_K$ (it is the only situation when $|C^{\downarrow L}| < 1$), and when $L = K$ then $|C| = 1$.

In addition to the projection, in this text we will need also the opposite operation which will be called a join. By a *join* of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ we will understand a set

$$A \otimes B = \{x \in \mathbf{X}_{K \cup L} : x^{\downarrow K} \in A \ \ \& \ \ x^{\downarrow L} \in B\}.$$

Notice that if K and L are disjoint then the join of the corresponding sets is just their Cartesian product

$$A \otimes B = A \times B.$$

For $K = L$, $A \otimes B = A \cap B$. If $K \cap L \neq \emptyset$ and $A^{\downarrow K \cap L} \cap B^{\downarrow K \cap L} = \emptyset$ then also $A \otimes B = \emptyset$.

In one of the following proofs we will need the following (rather technical) property of set joins.

Lemma 1. *Let $K_1 \cap K_2 \subseteq L \subseteq K_2 \subseteq N$. Then for any $C \subseteq \mathbf{X}_{K_1 \cup K_2}$ the following condition* (a) *holds if and only if both conditions* (b) *and* (c) *hold true.*

(a) $C = C^{\downarrow K_1} \otimes C^{\downarrow K_2}$;

(b) $C^{\downarrow K_1 \cup L} = C^{\downarrow K_1} \otimes C^{\downarrow L}$;

(c) $C = C^{\downarrow K_1 \cup L} \otimes C^{\downarrow K_2}$.

Proof. Let us prove the assertion in three steps. First, however, let us realize that

$$x \in C \implies \left(x^{\downarrow K_1} \in C^{\downarrow K_1} \ \& \ x^{\downarrow K_2} \in C^{\downarrow K_2}\right),$$

and therefore $C = C^{\downarrow K_1} \otimes C^{\downarrow K_2}$ is equivalent to

$$\forall x \in \mathbf{X}_{K_1 \cup K_2} \quad \left(x^{\downarrow K_1} \in C^{\downarrow K_1} \ \& \ \ x^{\downarrow K_2} \in C^{\downarrow K_2} \Longrightarrow x \in C\right).$$

(a) $\Longrightarrow$ (b).
Consider $x \in \mathbf{X}_{K_1 \cup L}$, such that $x^{\downarrow K_1} \in C^{\downarrow K_1}$ and $x^{\downarrow L} \in C^{\downarrow L}$. Since $x^{\downarrow L} \in C^{\downarrow L}$ there must exists (at least one) $y \in C^{\downarrow K_2}$, for which $y^{\downarrow L} = x^{\downarrow L}$. Now construct $z \in \mathbf{X}_{K_1 \cup K_2}$ for which $z^{\downarrow K_1} = x^{\downarrow K_1}$ and $z^{\downarrow K_2} = y$ (it is possible because $y^{\downarrow L} = x^{\downarrow L}$). From this construction we see that $z^{\downarrow K_1 \cup L} = x$. Therefore $z^{\downarrow K_1} = x^{\downarrow K_1} \in C^{\downarrow K_1}$ and $z^{\downarrow K_2} = y \in C^{\downarrow K_2}$ form which, because we assume that (a) holds, we get that $z \in C$, and therefore also $x = z^{\downarrow K_1 \cup L} \in C^{\downarrow K_1 \cup L}$.

(a) $\Longrightarrow$ (c).
Consider now $x \in \mathbf{X}_{K_1 \cup K_2}$, for which its projections $x^{\downarrow K_1 \cup L} \in C^{\downarrow K_1 \cup L}$ and $x^{\downarrow K_2} \in C^{\downarrow K_2}$. From $x^{\downarrow K_1 \cup L} \in C^{\downarrow K_1 \cup L}$ we immediately get that $x^{\downarrow K_1} \in C^{\downarrow K_1}$, which in combination with $x^{\downarrow K_2} \in C^{\downarrow K_2}$ (due to the assumption (a)) yields that $x \in C$.

(b) & (c) $\Longrightarrow$ (a).
Consider $x \in \mathbf{X}_{K_1 \cup K_2}$ such that $x^{\downarrow K_1} \in C^{\downarrow K_1}$ and $x^{\downarrow K_2} \in C^{\downarrow K_2}$. From the last property one gets also $x^{\downarrow L} \in C^{\downarrow L}$, which, in combination with $x^{\downarrow K_1} \in C^{\downarrow K_1}$ gives, because (b) holds true, that $x^{\downarrow K_1 \cup L} \in C^{\downarrow K_1 \cup L}$. And the last property in combination with $x^{\downarrow K_2} \in C^{\downarrow K_2}$ yields the required $x \in C$. ☐

Assignment notation

The role of a probability distribution from a probability theory is in Dempster-Shafer theory played by any of the set functions: belief function, plausibility function or basic (*probability or belief*) assignment. Knowing one of them, one can deduce the two remaining. In this paper we shall use exclusively basic assignments.

A *basic assignment* m on $\mathbf{X}_K$ $(K \subseteq N)$ is a function

$$m : \mathcal{P}(\mathbf{X}_K) \longrightarrow [0, 1],$$

for which

$$\sum_{\emptyset \neq A \subseteq \mathbf{X}_N} m(A) = 1.$$

For the sake of this paper it is reasonable to consider only normalized basic assignments, for which $m(\emptyset)$ equals always 0. If $m(A) > 0$, then A is said to be a *focal element* of m.

Having a basic assignment m on $\mathbf{X}_K$ one can consider its *marginal assignment* on $\mathbf{X}_L$ (for $L \subseteq K$), which is defined (for each $\emptyset \neq B \subseteq \mathbf{X}_L$):

$$m^{\downarrow L}(B) = \sum_{A \subseteq \mathbf{X}_K : A^{\downarrow L} = B} m(A).$$

Basic assignment m is said to be *Bayesian* if all its focal elements are *singletons*, i.e.

$$m(A) > 0 \implies |A| = 1.$$

In this case, namely, both the other two functions, belief Bel and plausibility Pl which are defined by the following formulas (for all $A \subseteq \mathbf{X}_K$)

$$Bel(A) = \sum_{B \subseteq A} m(A),$$
$$Pl(A) = 1 - Bel(\bar{A}),$$

are normalized additive functions, and therefore probability distributions.

Another special case is represented by simple basic assignments. Basic assignments m on $\mathbf{X}_K$ is called *simple* if there exists A ($\emptyset \neq A \subset \mathbf{X}_K$) and a positive number a such that $m(A) = a$ and $m(\mathbf{X}_K) = 1 - a$.

3 Operator of composition

Originally, the operator of composition was designed in probability theory as a tool enabling creation of multidimensional probability distributions - multidimensional models - by successive composition of low-dimensional distributions. The basic idea of this operator was simple. It generalized the fact that one can construct a 3-dimensional probability distribution $P(X,Y,Z)$ from two 2-dimensional ones $Q(X,Y)$ and $R(Y,Z)$ just by assigning

$$P(X,Y,Z) = Q(X,Y) \cdot R(Z|Y).$$

In this case P reflects all the information contained in Q, because evidently $P(X,Y) = Q(X,Y)$, and some of the information contained in R ($P(Z|Y) = R(Z|Y)$). Moreover, P does not contain any additional information, because for this probability distribution variables X and Z are conditionally independent given variable Y.

Introduction of the probabilistic operator of composition opened a study of a new area called compositional models, which was an alternative to Bayesian networks, or to Graphical Markov models in general. Though it appeared that Bayesian networks and compositional models described exactly the same class of probability distributions, study of a new type of models appeared useful. First of all it offered new points of view to multidimensional probability distribution representation. In addition to this, compositional models were in some situations more advantageous from the computational point of view (some of the marginal distributions, computation of which may be algorithmically rather expensive, were in a compositional model expressed explicitly).

Later, the operator of composition was designed and studied in possibility theory by Vejnarová [10]. Being inspired by Didier Dubois, we introduced the operator of composition also for basic assignments [5]; this definition is presented below. In that paper we also showed that if the operator of composition is applied to Bayesian basic assignments it usually yields the Bayesian basic assignment, which corresponds to the probability distribution, which is constructed by the probabilistic operator of composition from the respective probability distributions. The only exception from this situation occurs when composing basic assignments corresponding to probability distributions, for which their probabilistic composition is not defined. In such a case, result of composition of such Bayesian basic assignments is not Bayesian. In the next section we will reveal the main characteristics of such basic assignments.

Definition 1. For two arbitrary basic assignments m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ ($K \neq \emptyset \neq L$) a *composition* $m_1 \triangleright m_2$ is defined for each $C \subseteq \mathbf{X}_{K \cup L}$ by one of the following expressions:

[a] if $m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L}) > 0$ and $C = C^{\downarrow K} \otimes C^{\downarrow L}$ then

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L})};$$

[b] if $m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L}) = 0$ and $C = C^{\downarrow K} \times \mathbf{X}_{L \setminus K}$ then

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K});$$

[c] in all other cases $(m_1 \triangleright m_2)(C) = 0$.

Before illustrating the operator of composition on a simple example, let us remark that three expressions in Definition 1 correspond to three situations, which occur when one wants to define a basic assignments possessing those properties we highlighted when speaking about the probability distribution $P(X,Y,Z) = Q(X,Y) \cdot R(Z|Y)$. Point [a], in a way, directly corresponds to this well-known probabilistic formula. It disseminates the mass $m_1(C^{\downarrow K})$ into the respective subsets $C \subseteq \mathbf{X}_{K \cup L}$. The information describing the way how this mass is disseminated is taken over from m_2. Point [b] is applicable when

Table 1: 1-dimensional basic assignments m_1 and m_2.

$A \subseteq \mathbf{X}_1$	$m_1(A)$
$\{a\}$	0.5
$\{\bar{a}\}$	0.1
$\{a, \bar{a}\}$	0.4

$B \subseteq \mathbf{X}_2$	$m_2(B)$
$\{b\}$	0.5
$\{\bar{b}\}$	0.5

$m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) = 0$ and therefore m_2 does not determine the way how to disseminate the respective mass. Therefore the whole mass $m_1(C^{\downarrow K})$ is assigned to the least specific set: $C = C^{\downarrow K} \times \mathbf{X}_{L\setminus K}$ (expressing in this way maximal ignorance). Eventually, point [c] guarantees that no additional information is added to the resulting basic assignment $m_1 \triangleright m_2$. It assigns zero mass to all those subsets of $\mathbf{X}_{K\cup L}$, whose positive values would violate the notion of the required conditional independence (see e.g. [1]).

Example 1. Consider two 1-dimensional basic assignments[1] m_1, m_2 from Table 1, which are defined on $\mathbf{X}_1 = \{a, \bar{a}\}$ and $\mathbf{X}_2 = \{b, \bar{b}\}$, respectively.

Their composition $m_1 \triangleright m_2$ is in Table 2. Notice, that this composed basic assignment has only 6 focal elements, which means that for the remaining $(2^4 - 1) - 6 = 9$ subsets of $\mathbf{X}_1 \times \mathbf{X}_2$, values of $m_1 \triangleright m_2$ equal 0. It is the case of two groups of subsets. As for three subsets

$$\{ab, a\bar{b}\} = \{a\} \otimes \mathbf{X}_2,$$
$$\{\bar{a}b, \bar{a}\bar{b}\} = \{\bar{a}\} \otimes \mathbf{X}_2,$$
$$\{ab, a\bar{b}, \bar{a}b, \bar{a}\bar{b}\} = \mathbf{X}_1 \otimes \mathbf{X}_2,$$

their values of $m_1 \triangleright m_2$ are assigned by point [a] of Definition 1 and equal 0 because $m_2(\{b, \bar{b}\}) = 0$. On the other hand side, to the remaining six subsets

$$\{ab, \bar{a}\bar{b}\},$$
$$\{a\bar{b}, \bar{a}b\},$$
$$\{ab, a\bar{b}, \bar{a}b\},$$
$$\{ab, a\bar{b}, \bar{a}\bar{b}\},$$
$$\{ab, \bar{a}b, \bar{a}\bar{b}\},$$
$$\{a\bar{b}, \bar{a}b, \bar{a}\bar{b}\},$$

values of $m_1 \triangleright m_2$ are assigned by point [c] of Definition 1, because for these subsets it does not hold that $C = C^{\downarrow\{1\}} \otimes C^{\downarrow\{2\}}$. Assigning a positive value to any of these subsets we would, in a way, introduce a dependence of variables X_1 and X_2.

[1] In all examples in this paper we record in tables only focal elements. It means that for all subsets of space of discernment which are not included in the respective tables their respective basic assignment equals 0.

Table 2: Composed basic assignment $m_1 \triangleright m_2$.

$C \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$(m_1 \triangleright m_2)(C)$
$\{ab\}$	0.25
$\{a\bar{b}\}$	0.25
$\{\bar{a}b\}$	0.05
$\{\bar{a}\bar{b}\}$	0.05
$\{ab, \bar{a}b\}$	0.20
$\{a\bar{b}, \bar{a}\bar{b}\}$	0.20

Let us present the most important properties of the operator of composition for basic assignments.

Lemma 2. *Let $K, L \subseteq N$. For arbitrary basic assignments m_1, m_2 defined on $\mathbf{X}_K, \mathbf{X}_L$, respectively*

(i) *$m_1 \triangleright m_2$ is a basic assignment on $\mathbf{X}_{K\cup L}$;*

(ii) *$(m_1 \triangleright m_2)^{\downarrow K} = m_1$;*

(iii) *$m_1 \triangleright m_2 = m_2 \triangleright m_1 \iff m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L}$;*

(iv) *$L \supseteq M \supseteq (K \cap L)$*
$$\implies m_1 \triangleright m_2 = (m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2;$$

Proof. The first three properties were proved in [5]: properties (i)-(iii) are properties (i)-(iii) of Lemma 1. Thus, what has remained to be proved is just property (iv).

So, our goal is to show that for basic assignments m_1, m_2 and for any M such that $L \supseteq M \supseteq K \cap L$

$$(m_1 \triangleright m_2)(C) = ((m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2)(C).$$

holds true for any $C \subseteq \mathbf{X}_{K\cup L}$.

The proof will be performed in three steps corresponding to cases [a], [b], [c] of Definition 1.

Ad [a]. Assume that $C = C^{\downarrow K} \otimes C^{\downarrow L}$ and $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) > 0$. From this we get from Lemma 1 that also $C^{\downarrow K\cup M} = C^{\downarrow K} \otimes C^{\downarrow M}$, and therefore (since $K \cap L = K \cap M$)

$$(m_1 \triangleright m_2^{\downarrow M})(C^{\downarrow K\cup M}) = \frac{m_1(C^{\downarrow K}) \cdot m_2^{\downarrow M}(C^{\downarrow M})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})}.$$

In the rest of this step we have to distinguish two situations depending whether $m_2^{\downarrow M}(C^{\downarrow M})$ equals 0 or not.

If $m_2^{\downarrow M}(C^{\downarrow M}) > 0$ (realize that in this case also

$m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) > 0)$ then

$$
\begin{aligned}
((m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2)(C) \\
&= \frac{(m_1 \triangleright m_2^{\downarrow M})(C^{\downarrow K\cup M}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow M}(C^{\downarrow M})} \\
&= \frac{\frac{m_1(C^{\downarrow K})\cdot m_2^{\downarrow M}(C^{\downarrow M})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})} \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow M}(C^{\downarrow M})} \\
&= \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})} = (m_1 \triangleright m_2)(C).
\end{aligned}
$$

If $m_2^{\downarrow M}(C^{\downarrow M}) = 0$ then, according to Definition 1, either

$$((m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2)(C) = (m_1 \triangleright m_2^{\downarrow M})(C^{\downarrow K\cup M}),$$

in case that $C = C^{\downarrow K\cup M} \otimes \mathbf{X}_{L\setminus M}$, or

$$((m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2)(C) = 0,$$

in opposite case. However, in this case also

$$(m_1 \triangleright m_2^{\downarrow M})(C^{\downarrow K\cup M}) = \frac{m_1(C^{\downarrow K}) \cdot m_2^{\downarrow M}(C^{\downarrow M})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})} = 0,$$

and therefore $((m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2)(C) = 0$ regardless of the form of $C^{\downarrow L\setminus M}$ (i.e. for both situations: $C^{\downarrow L\setminus M} = \mathbf{X}_{L\setminus M}$ and $C^{\downarrow L\setminus M} \neq \mathbf{X}_{L\setminus M}$). Taking into consideration the fact that in the considered situation (i.e. $m_2^{\downarrow M}(C^{\downarrow M}) = 0$) also $m_2(C^{\downarrow L}) = 0$, and therefore also

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})} = 0,$$

we have finished the first step of the proof.

Ad [b]. Now we assume that $C = C^{\downarrow K} \otimes \mathbf{X}_{L\setminus K}$, and that $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) = 0$. In this case, naturally, also $m_2^{\downarrow M}(C^{\downarrow M}) = 0$ and $C = C^{\downarrow K} \otimes \mathbf{X}_{M\setminus K} \otimes \mathbf{X}_{L\setminus M}$. Therefore, according to case [b] of Definition 1,

$$(m_1 \triangleright m_2^{\downarrow M})(C^{\downarrow K\cup M}) = m_1(C^{\downarrow K}),$$

and because of the same reasons also

$$
\begin{aligned}
((m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2)(C) &= (m_1 \triangleright m_2^{\downarrow M})(C^{\downarrow K\cup M}) \\
&= m_1(C^{\downarrow K}).
\end{aligned}
$$

In this case also $(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K})$, and we have finished the second step of the proof.

Ad [c]. The last step is trivial. In this case, as the reader can immediately see, both $((m_1 \triangleright m_2^{\downarrow M}) \triangleright m_2)(C)$ and $(m_1 \triangleright m_2)(C)$ equal 0 and therefore they equal to each other. □

Table 3: 2-dimensional basic assignments m_3 and m_4.

$A \subseteq \mathbf{X}_{\{1,2\}}$	$m_3(A)$
$\{a\bar{b}\}$	0.5
$\{\bar{a}b\}$	0.1
$\{ab, \bar{a}b\}$	0.4

$B \subseteq \mathbf{X}_{\{2,3\}}$	$m_4(B)$
$\{bc\}$	0.5
$\{\bar{b}\bar{c}\}$	0.2
$\{b\bar{c}, \bar{b}c\}$	0.3

Table 4: Basic assignments $m_3 \triangleright m_4$ and $m_4 \triangleright m_3$.

$C \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$	$m_3 \triangleright m_4$	$m_4 \triangleright m_3$
$\{a\bar{b}\bar{c}\}$	0.5	0.2
$\{\bar{a}bc\}$	0.1	0.1
$\{abc, \bar{a}bc\}$	0.4	0.4
$\{ab\bar{c}, a\bar{b}c, \bar{a}b\bar{c}, \bar{a}\bar{b}c\}$		0.3

Example 2. Property (iii) of the previous lemma says that for consistent basic assignments the operator of composition is commutative. Since any couple of basic assignments defined on non-overlapping frames of discernment are consistent (because $m^{\downarrow\emptyset} = 1$), for basic assignments m_1 and m_2 from Table 1 $m_1 \triangleright m_2 = m_2 \triangleright m_1$. Therefore, if we want to illustrate non-commutativity of this operator we have to consider overlapping frames of discernment[2].

Consider basic assignments m_3, m_4 from Table 3. The reader can easily see that when computing $m_3 \triangleright m_4$, all the focal elements are computed according to case [a] of Definition 1. There are only three sets $C \subseteq \mathbf{X}_{\{1,2,3\}}$, for which $C = C^{\downarrow\{1,2\}} \otimes C^{\downarrow\{2,3\}}$, and for which both $m_3(C^{\downarrow\{1,2\}})$ and $m_3(C^{\downarrow\{2,3\}})$ are positive, namely

$$
\begin{aligned}
\{a\bar{b}\bar{c}\} &= \{a\bar{b}\} \otimes \{\bar{b}\bar{c}\}, \\
\{\bar{a}bc\} &= \{\bar{a}b\} \otimes \{bc\}, \\
\{abc, \bar{a}bc\} &= \{ab, \bar{a}b\} \otimes \{bc\}.
\end{aligned}
$$

On the other hand, when computing $m_4 \triangleright m_3$ there appears set $C = \{b\bar{c}, \bar{b}c\} \times \mathbf{X}_1$, for which $m_3(C^{\downarrow\{1,2\}}) = 0$ and therefore value $(m_4 \triangleright m_3)(C)$ is assigned by point [b] of Definition 1. The resulting basic assignment $m_4 \triangleright m_3$ is also recorded in Table 4.

Remark: In previous papers [5, 4] we showed a number of other properties of the operator of composition

[2]The simplest example of non-commutativity of the operator of composition can be got by considering two different assignments on the same frame of discernment. Then using property (i) of Lemma 2 we see that their composition is defined on the same frame of discernment as the considered original assignments and the non-commutativity of the operator $\triangleright$ immediately follows from property (ii) of Lemma 2.

for basic assignments, especially those useful for construction of multidimensional models. The four properties included in the previous lemma are those, which are sufficient to prove that conditional independence, if introduced with the help of the operator of composition (as done in Section 5), meets the semigraphoid axioms. In a way it is surprising that such a small group of elementary properties is sufficient. In connection with this fact a question arises whether the presented four properties are independent, whether some of them cannot be deduced from the remaining four.

Remark: Let us briefly answer a frequent question what is the relation of the introduced operator of composition and the famous Dempster's rule of combination[3]. Let us stress that the main difference emerges from the different purposes the operators where designed for. While Dempster's rule of combination was designed to have a tool enabling fusion of two basic assignments (the goal is to get a better information about the object than those contained in any of the original basic assignments), the operator of composition combines different descriptions of the object to comprehend all the information contained in original sources. This process corresponds to knowledge integration rather than knowledge fusion.

From the formal point of view this difference is reflected in property (ii) of Lemma 2, which holds for Dempster's rule of combination only in very specific (degenerated) situations. By the way, this difference is also the main reason why we consider the attempts to define a notion of conditional independence with the help of Dempster's rule of combination to be misleading.

4 Almost Bayesian basic assignments

One of the reasons (and from our point of view perhaps the most important) why D-S theory of evidence was designed and why it is in the center of attention of many researchers is the fact that probability theory has difficulties with representing some types of uncertainty; here we have in mind especially ignorance. For example, probability theory can hardly distinguish situation when an integer from $\{1, 2, \ldots, 6\}$ is determined by tossing a fair die, and when it is selected by a totally unknown mechanism (well, the second situation can be described by the set of all possible distributions, however it is rather inconvenient). On the other hand, D-S theory yields very complex models and the corresponding computational procedures are of extremely high algorithmic complexity. Now, we are about to specify a small family of basic assignments extending the set of Bayesian assignments but keeping the computational complexity on the level of probabilistic models. However, we have to admit that this new family, elements of which will be called almost Bayesian basic assignments, is very restrictive.

[3]Detailed study of formal similarities of these two operators will appear in [6].

Definition 2. Basic assignment m on $\mathbf{X}_K$ is called *cylindrical* if all its focal elements are point-cylinders.

Theorem 1. *Let $K, L \subseteq N$ and m_1, m_2 be basic assignments defined on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively. If m_1, m_2 are cylindrical then $m_1 \triangleright m_2$ is also cylindrical.*

Proof. To prove this assertion we have to realize that a projection $A^{\downarrow K}$ of a point-cylinder A is a point-cylinder. Moreover, join $A \otimes B$ of two point-cylinders A and B is again a point-cylinder (recall that $\emptyset$ is a point-cylinder).

Values of focal elements of basic assignment are computed according to either point [a] or point [b] of Definition 1. In case [a], a positive value can be assigned only if $C = C^{\downarrow K} \otimes C^{\downarrow L}$ and both $C^{\downarrow K}$ and $C^{\downarrow L}$ are point-cylinders. Case [b] is applied only when $C = C^{\downarrow K} \times \mathbf{X}_{L \setminus K}$. So in both cases positive value can be assigned only to point-cylinders. □

Definition 3. Basic assignment m on $\mathbf{X}_K$ is *sparse* if all its focal elements are pairwise disjoint.

Theorem 2. *Let $K, L \subseteq N$ and m_1, m_2 be basic assignments defined on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively. If m_1, m_2 are sparse then $m_1 \triangleright m_2$ is also sparse.*

Proof. Consider two non-disjoint focal elements C_1, C_2 of $m_1 \triangleright m_2$: $(m_1 \triangleright m_2)(C_1) > 0$ and $(m_1 \triangleright m_2)(C_2) > 0$. Since m_1 is marginal of $m_1 \triangleright m_2$, it is obvious that $C_1^{\downarrow K}$ and $C_2^{\downarrow K}$ are focal elements of m_1. Since we assume that C_1 and C_2 are non-disjoint the same must hold also for their projections

$$C_1^{\downarrow K} \cap C_2^{\downarrow K} \neq \emptyset$$

and therefore, because of our assumption that m_1 is sparse, $C_1^{\downarrow K} = C_2^{\downarrow K}$.

What are the focal elements C of $m_1 \triangleright m_2$, for which $C^{\downarrow K} = C_1^{\downarrow K}$? The answer to this question is offered by Definition 1 (realize that since we are considering focal elements C, values $(m_1 \triangleright m_2)(C)$ are defined by expressions in points [a] or [b]).

If $m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L}) > 0$ then the considered focal elements can be expressed in the form

$$C = C^{\downarrow K} \otimes C^{\downarrow L} = C_1^{\downarrow K} \otimes D,$$

Table 5: Sparse basic assignment m on $\mathbf{X}_{\{1,2\}}$.

$A \subseteq \mathbf{X}_1 \times \mathbf{X}_2$	$m(A)$
$\{ab\}$	0.2
$\{\bar{a}b\}$	0.3
$\{a\bar{b}, \bar{a}\bar{b}\}$	0.5

Table 6: Marginal basic assignments $m^{\downarrow\{1\}}$, $m^{\downarrow\{2\}}$.

$A \subseteq \mathbf{X}_1$	$m^{\downarrow\{1\}}(A)$
$\{a\}$	0.2
$\{\bar{a}\}$	0.3
$\{a, \bar{a}\}$	0.5

$B \subseteq \mathbf{X}_2$	$m^{\downarrow\{2\}}(B)$
$\{b\}$	0.5
$\{\bar{b}\}$	0.5

where $D \subseteq \mathbf{X}_L$ is a focal element of m_2 and $D^{\downarrow K\cap L} = C_1^{\downarrow K\cap L}$. From this one can immediately see that $C_1 = C_1^{\downarrow K} \otimes C_1^{\downarrow L}$ and $C_2 = C_1^{\downarrow K} \otimes C_2^{\downarrow L}$ are disjoint if and only if also focal elements $C_1^{\downarrow L}$ and $C_2^{\downarrow L}$of m_2 are disjoint. In our case, because m_2 is sparse, and because we assume that $C_1 \cap C_2 \neq \emptyset$, it means that $C_1^{\downarrow L} = C_2^{\downarrow L}$, and therefore also $C_1 = C_2$.

In case that $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) = 0$ then the situation is even simpler because in this case there can be only one focal element $C = C_1^{\downarrow K\cap L} \times \mathbf{X}_{L\setminus K}$, which means again that $C_1 = C_2$. □

Remark: It is not difficult to show that a marginal basic assignment of a cylindrical assignment is again cylindrical. However, it is important to realize that, as we illustrate in the following simple example, an analogous property for sparse basic assignments does not hold. Nevertheless, the main advantage of sparse basic assignments is the fact that the number of their focal elements does not exceed the cardinality of the respective frame of discernment, i.e. the number of probabilities necessary to define a general probability distribution.

Example 3. Consider 2-dimensional case with $\mathbf{X}_1 = \{a, \bar{a}\}$ and $\mathbf{X}_2 = \{b, \bar{b}\}$ and basic assignment m in Table 5. From Table 6 one can immediately see that while marginal basic assignment $m^{\downarrow\{2\}}$ is sparse, the other marginal assignment $m^{\downarrow\{1\}}$ is not.

Remark: Now we are ready to answer the question raised at the beginning of the previous section: what are the basic assignments which are obtained from Bayesian basic assignments by a multiple application of the operator of composition? Since all Bayesian assignments are obviously sparse and cylindrical, Theorems 1 and 2 guarantee that the basic assignments corresponding to compositional models from Bayesian basic assignments are also cylindrical and sparse. This fact, somehow, justifies the following definition.

Definition 4. Basic assignment is called *almost Bayesian* if it is sparse and cylindrical.

As said at the beginning of this section, an expressive power of almost Bayesian basic assignments is not too strong. For example, even non-degenerated simple basic assignments are not almost Bayesian. Roughly speaking: Having a Bayesian basic assignment one knows a probability of each point of the frame of discernment. Having an almost Bayesian basic assignment and a fixed point of the frame of discernment one either knows its probability, or knows that it belongs to a cylindrical subset of the frame of discernment among whose elements one cannot make a difference; she knows only the probability of the whole subset. Nevertheless, let us stress once more that the importance of almost Bayesian assignments is in the fact that they describe compositional models constructed from an arbitrary system of low-dimensional probability distributions, which means that even in situations when probabilistic operator of composition is not defined. In this way we are getting a slight extension of probability theory.

5 Conditional independence

In this paper our attention is concentrated on properties of basic assignments which are, in a way, promising from the point of view of computational complexity. Last section was devoted to almost Bayesian basic assignment whose number of focal elements is not higher than the number of probabilities by which a general probability distribution must be specified.

It is well known that efficiency of Bayesian models is based on making the best of the dependence structure of the model, i.e. taking advantage of the knowledge of conditional independence relations [8, 9] holding for the multidimensional distribution in question. This is because the notion of conditional independence in probability theory is equivalent to the notion of *factorization*: for probability distribution P variables X and Z are conditionally independent given variable Y *iff* distribution $P(X, Y, Z)$ is uniquely determined by its marginals $P(X, Y)$ and $P(Y, Z)$. Unfortunately, as shown by Studený [8, 1], the notion of conditional non-interactivity (Shenoy's factorization [7], Studený conditional independence [8]) presented in [1] is *not consistent with marginalization*: there are situations when for two consistent basic assignments there does not exist their common extension with the respective conditional non-interactivity (for more precise explanation see footnote no. 6).

Therefore, in this paper we are going to eliminate this drawback using the definition of conditional independence for basic assignments introduced in [4], which is in fact based on the notion of factorization. Moreover we will present new proofs showing that for this concept all the semigraphoid axioms hold true. These proofs will be based on the fundamental properties of the operator of composition presented in Lemma 2. It should be stressed that the novelty of these proofs is mainly in application of property (iv) of Lemma 2, which seems to be surprisingly weak (and which, in a way, extends property (ii) of the same lemma).

Let us consider an arbitrary basic assignment. We will say that two groups of variables are conditionally independent given the third group of variables if the respective marginal basic assignment can be decomposed (factorized) in the way that it can be expressed as a composition of its respective smaller marginal assignments. Precisely this notion is introduced in the following definition.

Definition 5. Consider a basic assignment m on $\mathbf{X}_N$ and three disjoint index sets $K, L, M \subset N$, $K \neq \emptyset \neq L$. We say that groups of variables X_K and X_L are *conditionally independent given variables* X_M if

$$m^{\downarrow K\cup L\cup M} = m^{\downarrow K\cup M} \triangleright m^{\downarrow L\cup M}.$$

In symbol this fact will be recorded $K \perp\!\!\!\perp_m L \,|\, M$.

Example 4. Consider a basic assignment m on the same 3-dimensional binary frame of discernment as in previous examples: $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$. If variables X_1 and X_2 are independent, i.e. $1 \perp\!\!\!\perp_m 2$, from Definition 1 one can immediately see that for all focal elements $C \subseteq \mathbf{X}_1 \times \mathbf{X}_2$ of the 2-dimensional marginal $m^{\downarrow\{1,2\}}$ it holds that $C = C^{\downarrow\{1\}} \otimes C^{\downarrow\{2\}}$. It means that from all 15 non-empty subsets of $\mathbf{X}_1 \times \mathbf{X}_2$ only 9 of them are potential focal elements (six subsets of $\mathbf{X}_1 \times \mathbf{X}_2$ that cannot be focal elements are listed in Example 1). Naturally, this condition on focal elements is only a necessary condition for the independence. This condition is not sufficient. For example, the reader can easily check that the two basic assignments $m_1 \triangleright m_2$ from Table 2 and m_3 from Table 3 (both defined on $\mathbf{X}_1 \times \mathbf{X}_2$) have the same marginal assignments: $((m_1 \triangleright m_2)^{\downarrow\{1\}} = m_3^{\downarrow\{1\}} = m_1$ and $(m_1 \triangleright m_2)^{\downarrow\{2\}} = m_3^{\downarrow\{2\}} = m_2)$. Moreover, for all of their focal elements the required property $C = C^{\downarrow\{1\}} \otimes C^{\downarrow\{2\}}$ holds true and simultaneously

$$1 \perp\!\!\!\perp_{m_1 \triangleright m_2} 2 \quad \text{and} \quad 1 \not\perp\!\!\!\perp_{m_3} 2.$$

Analogously to what has just been said about (unconditional) independence, there is a necessary condition also on focal elements of basic assignments with conditional independence. Conditional independence

$$1 \perp\!\!\!\perp_m 3 \,|\, 2$$

means that all focal elements $C \subseteq \mathbf{X}_{\{1,2,3\}}$ of m must be of the form

$$C = C^{\downarrow\{1,2\}} \otimes C^{\downarrow\{2,3\}}.$$

It is not difficult to show that this property holds true only for 99 out of all possible 255 nonempty subsets of $\mathbf{X}_{\{1,2,3\}}$.

In the rest of this section we will show that the ternary relation $K \perp\!\!\!\perp_m L \,|\, M$ is a *semigraphoid*, i.e. it meets the four semigraphoid axioms listed below. For this, we will exclusively use the properties of the operator of composition presented in Lemma 2. In what follows, each axiom is reformulated into the language of composition and the corresponding theorem is proved.

Symmetry

$I \perp\!\!\!\perp_m J \,|\, L \implies J \perp\!\!\!\perp_m I \,|\, L$

Theorem 3. *If* $m^{\downarrow I\cup J\cup L} = m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup L}$ *then also* $m^{\downarrow I\cup J\cup L} = m^{\downarrow J\cup L} \triangleright m^{\downarrow I\cup L}$.

Proof. The assertion follows immediately from the fact that marginals $m^{\downarrow I\cup L}$ and $m^{\downarrow J\cup L}$ are consistent, and therefore property (iii) may be applied

$$m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup L} = m^{\downarrow J\cup L} \triangleright m^{\downarrow I\cup L}.$$

□

Decomposition

$I \perp\!\!\!\perp_m J \cup K \,|\, L \implies I \perp\!\!\!\perp_m K \,|\, L$

Theorem 4. *If* $m^{\downarrow I\cup J\cup K\cup L} = m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup K\cup L}$ *then also* $m^{\downarrow I\cup K\cup L} = m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}$.

Proof. The assertion will be obtained just by application of properties (iv) and (ii)

$$\begin{aligned}
m^{\downarrow I\cup K\cup L} &= \left(m^{\downarrow I\cup J\cup K\cup L}\right)^{\downarrow I\cup K\cup L} \\
&= \left(m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup K\cup L}\right)^{\downarrow I\cup K\cup L} \\
&= \left((m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}) \triangleright m^{\downarrow J\cup K\cup L}\right)^{\downarrow I\cup K\cup L} \\
&= m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}.
\end{aligned}$$

□

Weak Union

$I \perp\!\!\!\perp_m J \cup K \,|\, L \implies I \perp\!\!\!\perp_m J \,|\, K \cup L$

Theorem 5. *If* $m^{\downarrow I\cup J\cup K\cup L} = m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup K\cup L}$ *then also* $m^{\downarrow I\cup J\cup K\cup L} = m^{\downarrow I\cup K\cup L} \triangleright m^{\downarrow J\cup K\cup L}$.

Proof. To prove this assertion we have to realize that, due to property (iv),

$$m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup K\cup L} = \left(m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}\right) \triangleright m^{\downarrow J\cup K\cup L},$$

and that, because the assumptions of Theorem 4 are fulfilled, also

$$m^{\downarrow I\cup K\cup L} = m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}.$$

Using these two equalities we finish the proof in a simple way

$$\begin{aligned} m^{\downarrow I\cup J\cup K\cup L} &= m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup K\cup L} \\ &= \left(m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}\right) \triangleright m^{\downarrow J\cup K\cup L} \\ &= m^{\downarrow I\cup K\cup L} \triangleright m^{\downarrow J\cup K\cup L}. \end{aligned}$$

□

Contraction

$I \perp\!\!\!\perp_m K \,|\, L \;\&\; I \perp\!\!\!\perp_m J \,|\, K \cup L \implies I \perp\!\!\!\perp_m J \cup K \,|\, L$

Theorem 6. *If* $m^{\downarrow I\cup K\cup L} = m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}$, *and* $m^{\downarrow I\cup J\cup K\cup L} = m^{\downarrow I\cup K\cup L} \triangleright m^{\downarrow J\cup K\cup L}$, *then also* $m^{\downarrow I\cup J\cup K\cup L} = m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup K\cup L}$.

Proof. We will follow the same idea as in the preceding proof but in the reverse order. First, we will use property (iv) and then both assumptions of this assertion.

$$\begin{aligned} m^{\downarrow I\cup L} \triangleright m^{\downarrow J\cup K\cup L} &= \left(m^{\downarrow I\cup L} \triangleright m^{\downarrow K\cup L}\right) \triangleright m^{\downarrow J\cup K\cup L} \\ &= m^{\downarrow I\cup K\cup L} \triangleright m^{\downarrow J\cup K\cup L} \\ &= m^{\downarrow I\cup J\cup K\cup L}. \end{aligned}$$

□

6 Conclusions

In the paper we dealt with the two problems connected with computational complexity of Dempster-Shafer theory of evidence. Since full generality of the models leads to exponential grows of space and computational complexity we showed that focusing our attention only to models, which are constructed from Bayesian basic assignments by application of the operator of composition, one does not get beyond the boundaries of a rather limited class of models, which are called in the paper almost Bayesian. The most advantageous characteristics of these models is the fact that though they are able to describe a special type of an ignorance, they do not have a higher space requirements than classical probabilistic models.

The other goal of this paper was to show that when accepting the notion of conditional independence based on factorization corresponding to the operator of composition, one can easily prove validity of semigraphoid axioms just with the help of the four very elementary properties from Lemma 2. Since the same idea was employed by Prakash P. Shenoy in [7], a very natural question arises: what is the relation of composition introduced in this paper and the Shenoy's notion of *combination*?

Looking at Shenoy axioms[4] C1, C2 and C3 we see that Shenoy's axiom C1 *(Domain)* is equivalent to property (i) of Lemma 2 and therefore it holds also for our composition. However Shenoy's axioms C2 (Associative) and C3 *(Commutative)* hold for composition only under special conditions. The operator of composition is commutative only for consistent basic assignments; point (iii) of Lemma 2. In definition of conditional independence (Definition 5 of this paper) we consider only composition of consistent assignments (marginals of the considered basic assignment) and therefore we were able to prove axiom of Symmetry. Nevertheless, associativity holds for the operator of composition only under very specific conditions[5] and therefore the Shenoy's proofs cannot be used. Moreover, property (ii) of Lemma 2 does not hold for Shenoy's combination. So, one cannot be surprised that both of the definitions of conditional independence (i.e. the one proposed in this paper and Shenoy's conditional independence following from the definitions in Section 5 of [7]) are different from each other. They coincide only for unconditional independence and for conditional independence in case of Bayesian basic assignments. Moreover, as we showed in [4], our concept of conditional independence does not suffer from the drawback described in detail in [1], where the authors show that the notion of conditional independence used by Shenoy is not *consistent with marginalization*[6]. Therefore, we can conclude that our concept of conditional independence seems to meet better some of the intuitive requirements. Nevertheless, a question what is the relation of this notion and concepts of conditional basic assignments remains still open.

[4] We do not comment axiom C4 *(Zero)* because we consider only normalized basic assignments.

[5] For example, for basic assignments m_1, m_2, m_3 defined on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}, \mathbf{X}_{K_3}$, respectively

$$K_1 \supseteq (K_2 \cap K_3) \implies (m_1 \triangleright m_2) \triangleright m_3 = m_1 \triangleright (m_2 \triangleright m_3).$$

[6] Roughly speaking: one can find two consistent basic assignments m_1, m_2, on $\mathbf{X}_1 \times \mathbf{X}_2$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively, for which there does not exist a 3-dimensional basic assignment m on $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ having m_1 and m_2 as its marginals, and for which $1 \perp\!\!\!\perp_m 3 \,|\, 2$.

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