Sets of Desirable Gambles and Credal Sets

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Gambles

- We have an uncertain taking values on a finite set Ω
- A gamble is a mapping $X : \Omega \to \mathbb{R}$
- $X(\omega)$ is the reward if $X = \omega$
- Some gambles are clearly desirable for us (for example if $X(\omega) > 0, \forall\omega$) and other are undesirable (for example if $X(\omega) < 0, \forall\omega$)

Example

- Consider the result of football match with $\Omega = \{0-0, 1-0, 0-1, 1-1, 2-0, ..., 15-15\}$
- A gamble $X_1(1-0) = 10, X_1(r) = -1,$ otherwise
- Another example could be $X_2(i-j) = 1,$ if $i > j$, $X_2(i-j) = -1,$ if $i < j$ and 0 otherwise.
- If we believe in 'draw' we could accept: $X_3(i-i) = 1, \quad X_3(i-j) = -1, i \neq j$

Coherent Set of Desirable Gambles

D1. $0 \notin \mathcal{D}$,
D2. if $X \in \mathcal{L}$ and $X > 0$ then $X \in \mathcal{D}$,
D3. if $X \in \mathcal{D}$ and $c \in \mathbb{R}^+$ then $cX \in \mathcal{D}$,
D4. if $X \in \mathcal{D}$ and $Y \in \mathcal{D}$ then $X + Y \in \mathcal{D}$.

Basic Consistency Condition

A set of desirable gambles $\mathcal{D}$ avoids partial loss if and only if $0 \notin \mathcal{D}$

We should not accept: $f(i - j) = -1$ if $i > j$ and 0 otherwise.

Closed Set of Gambles

A set of desirable gambles $\mathcal{D}$ is closed if D2, D3, and D4 are verified.

Almost Desirable Gambles

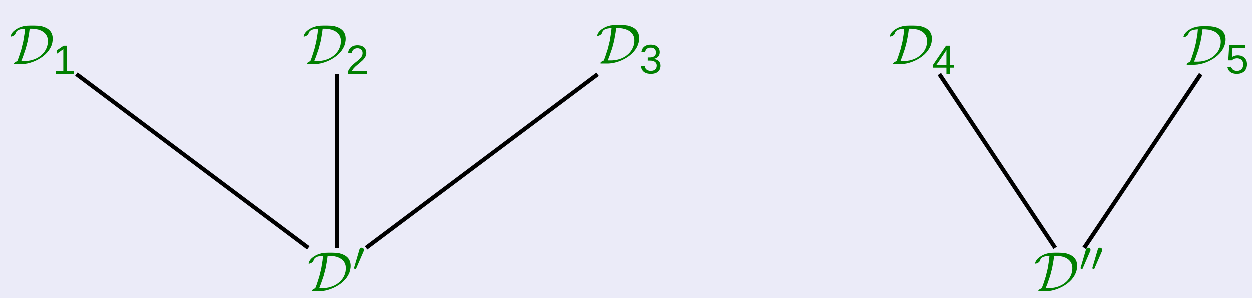
D1' $\forall X \in \mathcal{D}^*$, we have $\sup X \geq 0$
D2 If $X > 0$, then $X \in \mathcal{D}^*$
D3 If $X \in \mathcal{D}^*$ and $\lambda > 0$ then $\lambda . X \in \mathcal{D}^*$
D4 If $X_1, X_2 \in \mathcal{D}^*$ then $X_1 + X_2 \in \mathcal{D}^*$
D5 If $X + \epsilon \in \mathcal{D}^*, \quad \forall \epsilon > 0$ then $X \in \mathcal{D}^*$

Basic Consistency Condition

A set of almost desirable gambles $\mathcal{D}^*$ avoids sure loss if and only if $\forall X \in \mathcal{D}$ such that $\sup X \geq 0$

Desirable vs Almost Desirable Gambles

Desirable Gambles



Almost Desirable Gambles

Desirable Gambles are a more general model

Desirable vs. almost desirable gambles

Let us consider the gambles:
$X_\epsilon(i-j) = \epsilon$ if $i - j \neq 15 - 15$, $\quad X_\epsilon(15-15) = -1$

- It is possible that all the gambles X_ϵ are desirable.
- If they are almost desirable, then the gamble:
  $X_0(i-j) = 0$ if $i - j \neq 15 - 15$, $\quad X_0(15-15) = -1$
  is almost desirable.
- Almost desirable gambles avoids uniform loss, but not partial loss.

Strictly Desirable Gambles

D2 If $X > 0$, then $\mathcal{D}$

D3 If $X \in \mathcal{D}$ and $\lambda > 0$ then $\lambda.X \in \mathcal{D}$

D4 If $X_1, X_2 \in \mathcal{D}$ then $X_1 + X_2 \in \mathcal{D}$

D5' If $X \in \mathcal{D}$ then either $X > 0$ or $\exists \epsilon > 0$, $X - \epsilon \in \mathcal{D}$

Basic Consistency Condition: A set of desirable gambles $\mathcal{D}$ avoids partial loss ($0 \notin \mathcal{D}$)

Upper and Lower Previsions and Desirable Gambles

- The lower prevision of gamble X is
  $\underline{P}(X) = \sup\{\alpha \ : \ X - \alpha \in \mathcal{D}\}$
  The supremum of the buying prices.
- The upper prevision of gamble X is
  $\overline{P}(X) = \inf\{\alpha \ : \ -X + \alpha \in \mathcal{D}\}$
  The infimum of the selling prices.

Credal Sets and Desirable Gambles

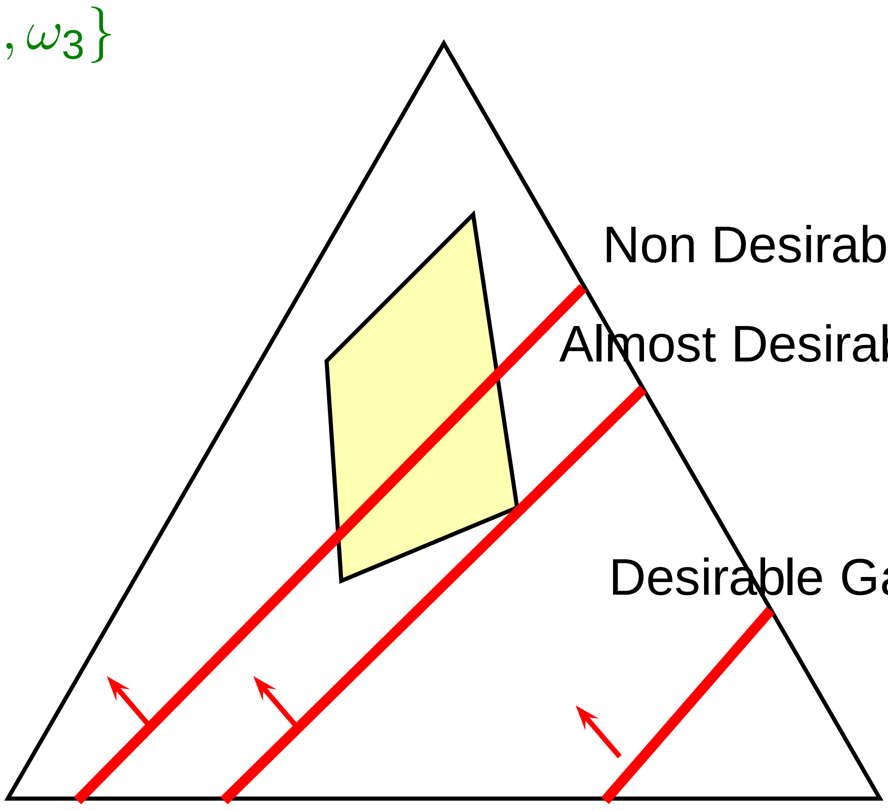
- A set of desirable gambles $\mathcal{D}$ defines a credal set:
  $\mathcal{P}_\mathcal{D} = \{P \ : \ P[X] \geq 0, \forall X \in \mathcal{D}\}$
- A set of desirable gambles $\mathcal{D}$ and the set of almost desirable gambles $\mathcal{D}^*$ define the same credal set
- A credal set $\mathcal{P}$ defines a set of almost desirable gambles:
  $\mathcal{D}^*_\mathcal{P} = \{X \ : \ P[X] \geq 0, \forall P \in \mathcal{P}\}$
- But several sets of desirable gambles can be associated:
  $\mathcal{D}_\mathcal{P} = \{X \ : \ P[X] > 0, \forall P \in \mathcal{P}\} \cup \{X \ : \ X > 0\}$

$$\mathcal{D}'' = \{X \ : \ P[X] \geq 0, \forall P \in \mathcal{P}, \quad \exists P \in \mathcal{P} P[X] > 0\} \cup \{X \ : \ X > 0\}$$

Graphical Representation: Credal Set

$$E_P[X] \geq 0, \forall P \in \mathcal{P}$$

$\Omega = \{\omega_1, \omega_2, \omega_3\}$



Conditioning

If we have a set of desirable gambles $\mathcal{D}$ and we observe event B, the conditional set of desirable gambles given B is given by:

$$\mathcal{D}_{\mathcal{B}} = \{X \ : \ X.I_B \in \mathcal{D}\} \cup \{X \ : \ X > 0\}$$

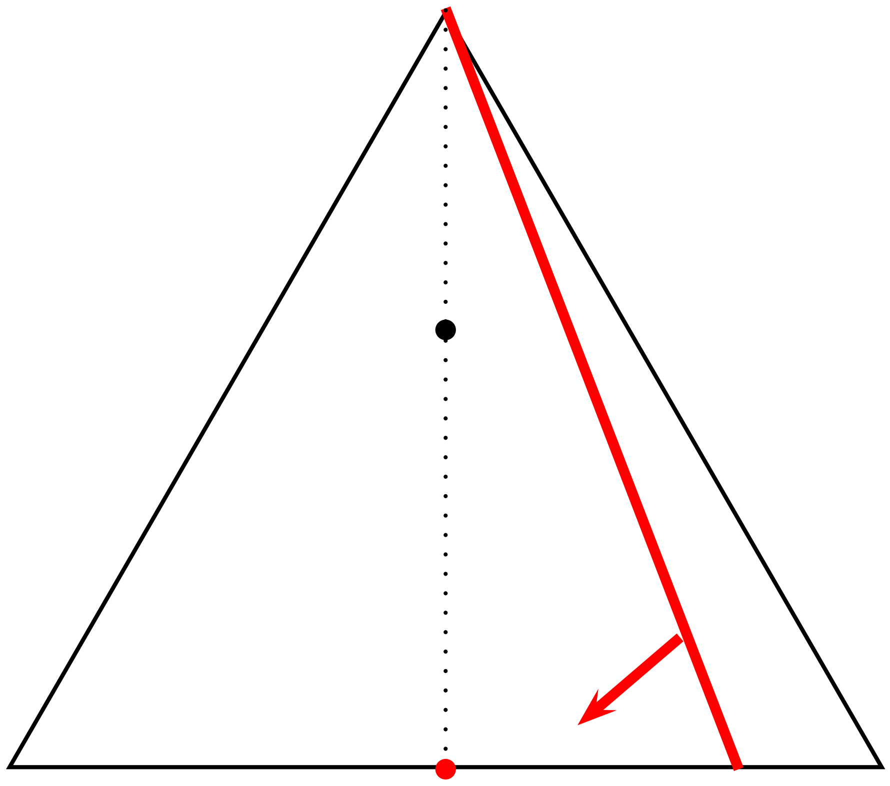
Example

I we accept a gamble
$X(Win) = 1, \quad X(Loss) = -1, \quad X(Draw) = 0,$
if we know that $Draw$ has not happened, then we should accept any gamble:
$Y(Win) = 1, \quad Y(Loss) = -1, \quad Y(Draw) = \alpha$

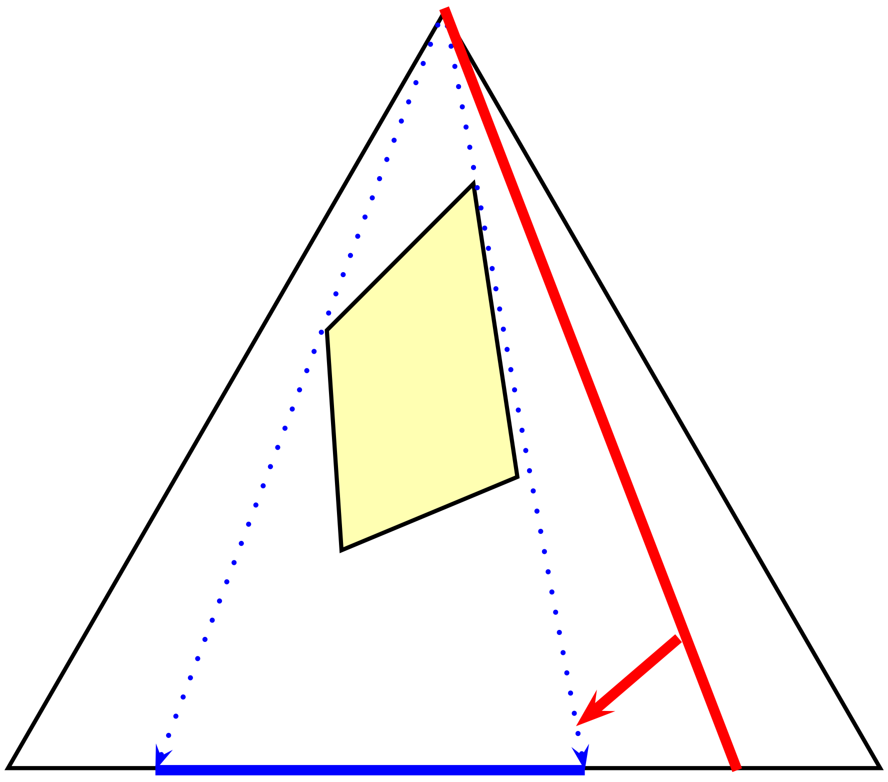
In fact, all the conditional information is in $\mathcal{D}$.

Conditioning

$\Omega = \{Draw, Win, Loss\}$



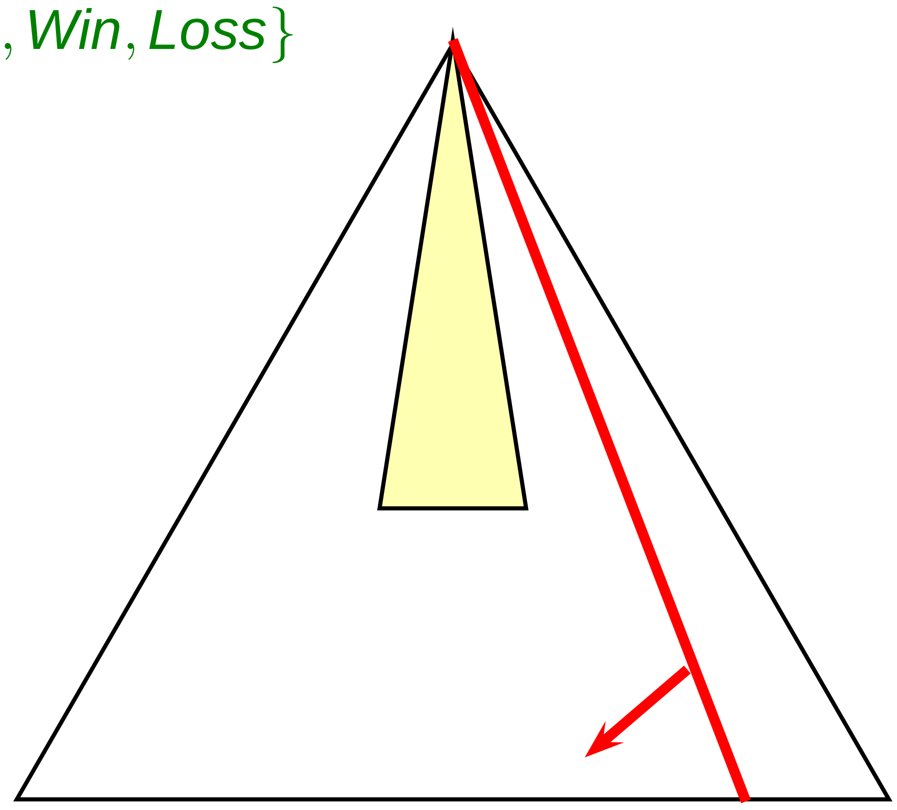
$B = \{Win, Loss\}$



If $\underline{P}(B) > 0$, then the credal set associated to the conditional set $\mathcal{D}$ is uniquely determined with independence of what happens with gambles in the frontier.

Conditioning: Lower Probability equal to 0

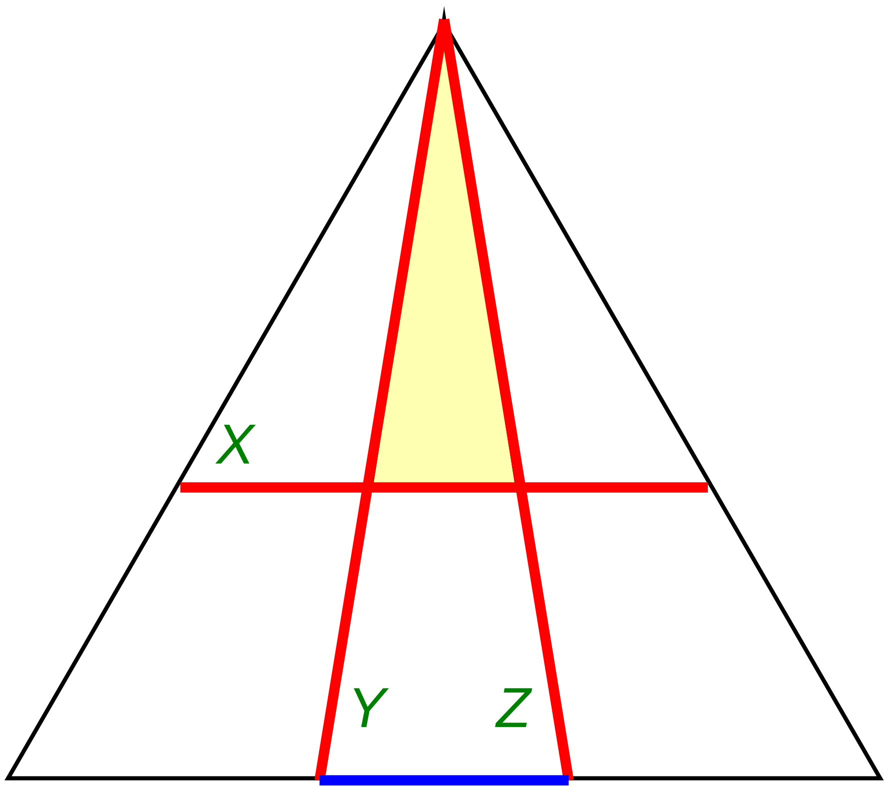
$\Omega = \{Draw, Win, Loss\}$

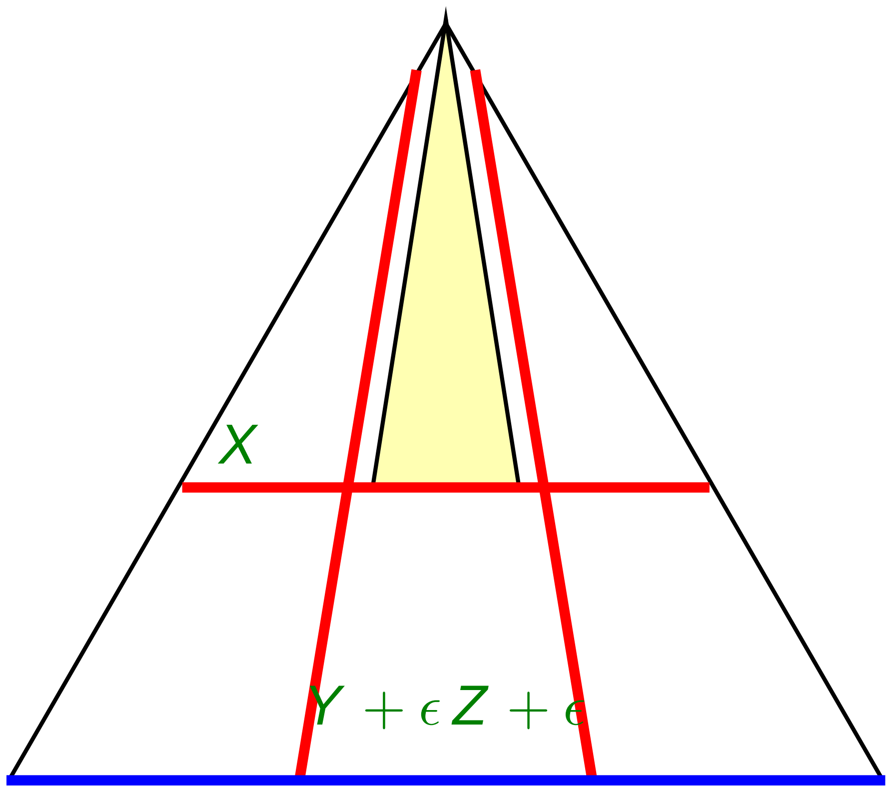


$B = \{Win, Loss\}$

If $\underline{P}(B) = 0$, all the gambles with $X(D) = 0.0$ are in the frontier. The credal set does not contains information about the conditioning.

Conditioning: Lower Probability equal to 0





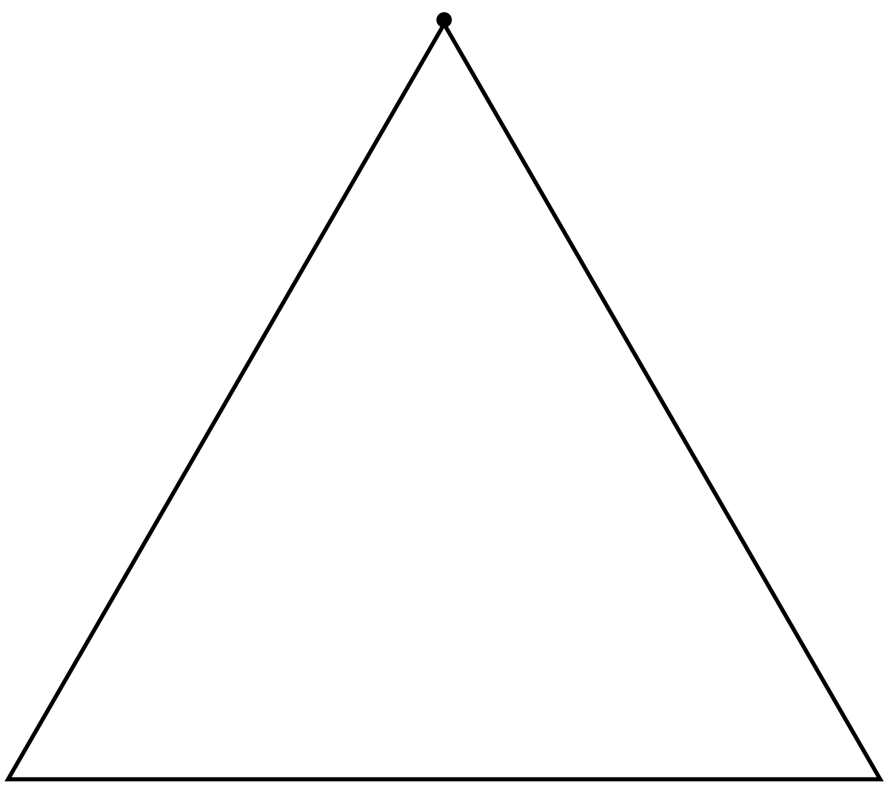
$B = \{Win, Loss\}$
This situation is compatible with accepting as desirable the gambles:

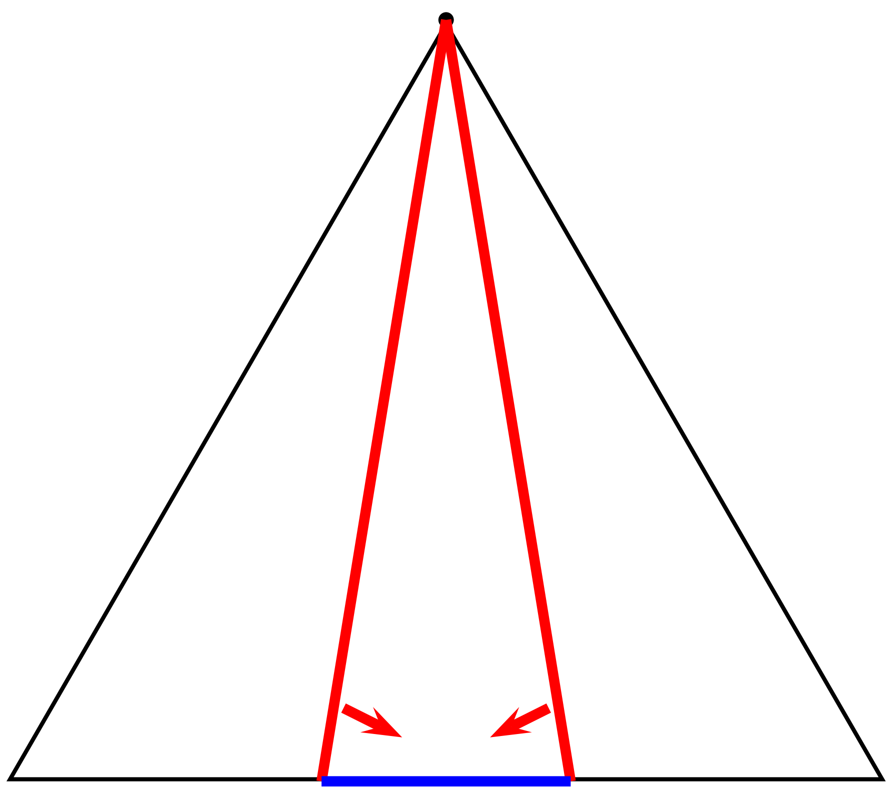
$$\begin{array}{lll} X(D)=1, & X(W)=-1, & X(L)=-1 \\ Y(D)=0, & Y(W)=1.2, & Y(L)=-1 \\ Z(D)=0, & Z(W)=-1, & Z(L)=1.2 \end{array}$$

But it is also compatible with gambles $\{X, Y+\epsilon, Z+\epsilon\}$
In this case, the conditioning is very wide: natural extension.

The case $\overline{P}(B) = 0$

- Imagine that we have $\omega_1 =$ *'There are less than 30 goals'*; $\omega_2 =$ *'Win or Draw with 30 goals or more in total'*; $\omega_3 =$ *'Loss with 30 goals or more in total'*.
- It is possible that we accept any gamble with $X(\omega_1)=\epsilon, \quad X(\omega_2)=-1, \quad X(\omega_3)=-1$
- If $B=\{\omega_2,\omega_3\}$, $\overline{P}(B)=\underline{P}(B)=0$.
- The conditioning will depend of which gambles $g(\omega_1)=0, \quad g(\omega_2)=\alpha_1, \quad g(\omega_3)=\alpha_2$





Regular Extension

- I have an urn with *Red*, *Blue*, *White* balls.
- I know that there is exactly the same number of Blue and White balls.
- This situation can be represented by the convex set of probability distributions:

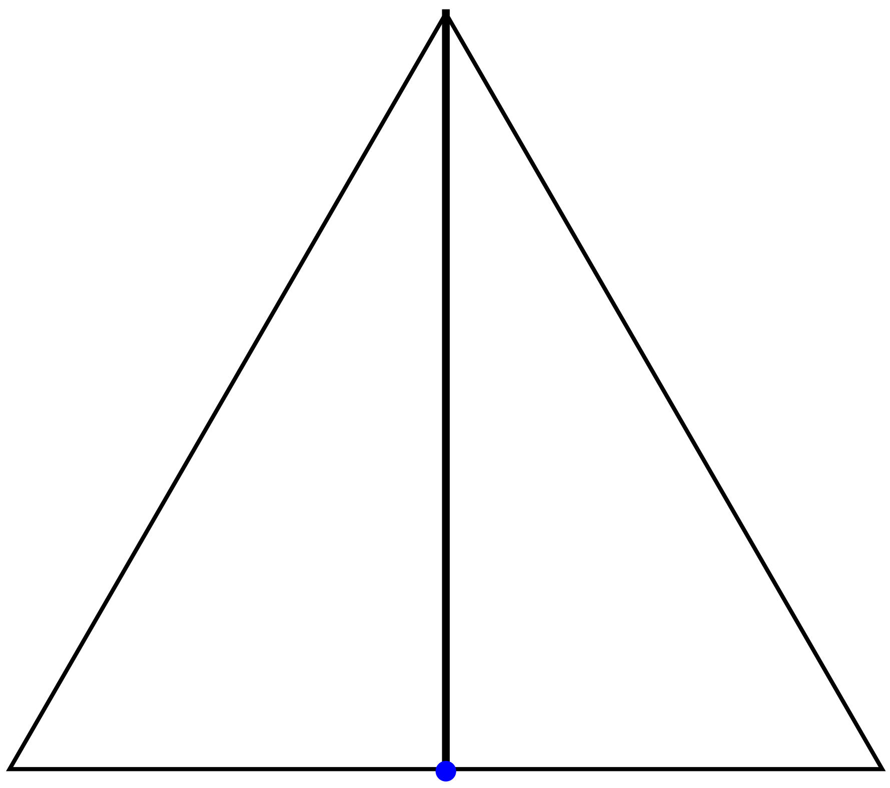
	Red	*Blue*	*White*
P_1	1	0	0
P_2	0	0.5	0.5

- If the set of desirable gambles is:

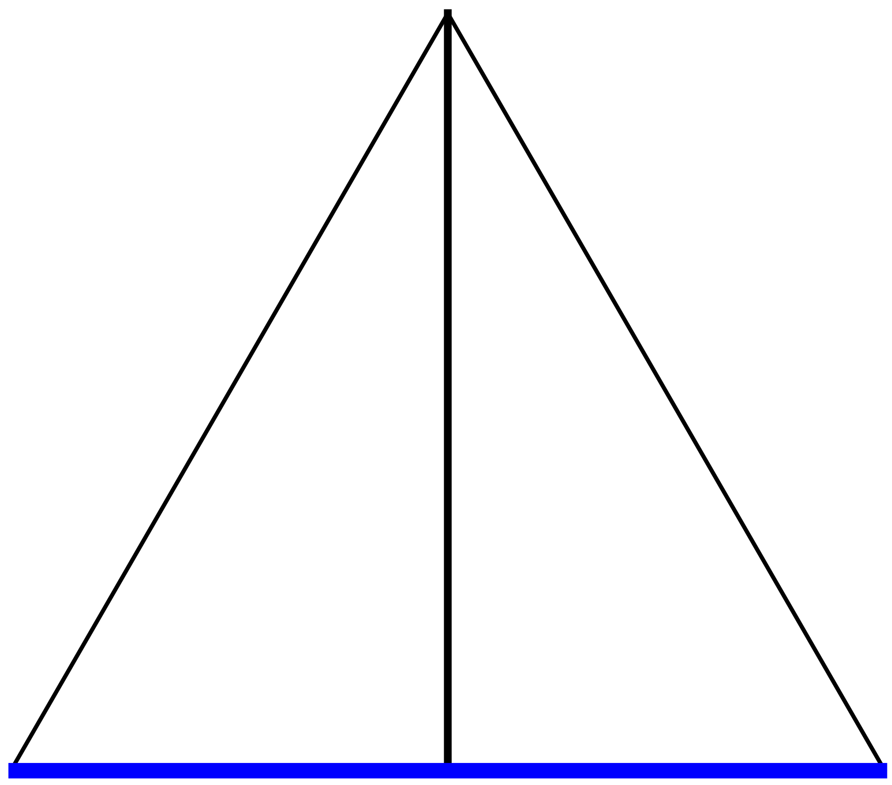
$$\mathcal{D}' = \{X \,:\, E_P[X] > 0, \forall P \in \mathcal{P}\}$$

then, if we know that a ball randomly selected from the urn is not red, then conditional to this information, the gamble $X(Blue) = 2, \quad X(White) = -1$ is not accepted.

- This does not seem reasonable. I should accept any gamble in which $X(Blue) + X(White) > 0$.



Regular Extension



Natural Extension

Regular Extension

$$\begin{tikzcd}[column sep=huge, row sep=huge] \mathcal{D} \arrow[r, "\text{Credal Set}"] \arrow[d, "\text{Conditioning}"'] & \mathcal{P}_{\mathcal{D}} \arrow[d, "\text{Regular Conditioning}"'] \\ \mathcal{D}_B \arrow[r, "\text{Credal Set}"] & \mathcal{P}_{\mathcal{D}_B} \end{tikzcd}$$

Theorem

Desirable gambles, regular extension is obtained assuming if $\underline{P}(B) > 0$ or:

$$X \in \mathcal{D}^* \text{ and } -X \notin \mathcal{D}^* \Rightarrow X \in \mathcal{D}.$$

Natural Extension - Encoding sets of gambles

Natural Extension

If $\mathcal{F}$ is a set of gambles, its natural extension $\overline{\mathcal{F}}$ is the set of gambles obtained from $\mathcal{F}$ applying axioms A2, A3, and A4 (the minimum set of gambles containing $\mathcal{F}$ and verifying these axioms.

Finitely Generated Sets of Gambles

A set of almost desirable gambles $\mathcal{D}$ is finitely generated if $\mathcal{D} = \overline{\mathcal{D}}_0$ where $\mathcal{D}_0$ is finite.

This definition is not appropriate for desirable gambles. We could not represent $P(B) = 0$. Which is equivalent to the acceptance of gambles $\epsilon.I_{B^c} - I_B$ for any ϵ.

Basic Reasoning Tasks

1. to determine whether the natural extension $\overline{\mathcal{F}}$ is coherent (i.e. $0 \notin \overline{\mathcal{F}}$),
2. given X, to determine whether $X \in \overline{\mathcal{F}}$,
3. given X and $B \subset \Omega$, to compute $\underline{P}(X|B)$ and $\overline{P}(X|B)$ under $\overline{\mathcal{F}}$ when this set is coherent.

Theorem

If $\mathcal{F}$ is an arbitrary set of gambles such that $\overline{\mathcal{F}}$ is coherent, then $X \in \overline{\mathcal{F}}$ if and only if $\overline{\mathcal{F} \cup \{-X\}}$ is not coherent.

ϵ-set representation

A basic set of gambles is a set of gambles $\mathcal{F}_{X,B} = \{X + \epsilon B \,:\, \epsilon > 0\}$, where X is an arbitrary gamble and $B \subseteq \Omega$, denoted as (X, B).
ϵ-set representation: $\mathcal{F}$ the union of: $(X_1, B_1), \dots, (X_k, B_k)$

Representation of Conditional Probabilities

$\underline{P}(X|B) = c$ is represented by means of $((X - c)B, B)$
$\overline{P}(X|B) = c$ is represented by means of $((c - X)B, B)$

Checking Consistency

$\mathcal{F}$ generated by $(X_1, B_1), \dots, (X_k, B_k)$: system in λ_i and ϵ has no solution:

$$\begin{array}{l} \sum_{i=1}^{k} \lambda_i (X_i + \epsilon B_i) \leq 0 \\ \lambda_i \geq 0, \quad \epsilon > 0 \end{array}$$

Algorithms in P. Walley, R. Pelessoni, P. Vicig (2004).

1. Set $I = \{1, \dots, k\}$
2. Solve
   $\sup \sum_i \tau_i$
   s.t. $\sum_i (\lambda_i X_i + \tau_i . B_i) \leq 0$
   $\lambda_i \geq 0, \quad 0 \leq \tau_i \leq 1$
3. Let $I' = \{i \mid \tau_i = 1\} >$ in the optimal solution
4. If $I' = \emptyset$, then Return(Consistency)
4. If $I' = I \neq \emptyset$ then Return(Nonconsistency)
5. else $I = I'$ and goto 2

To compute $\underline{P}(X|B)$

$$\begin{array}{l} \sup \alpha \\ s.t. \\ \sum_{i=1}^{k} \lambda_i (X_i + \epsilon B_i) \leq (X - \alpha) B \\ \epsilon > 0, \lambda_i \geq 0 \end{array}$$

Maximal Sets of Gambles

Definition

We will say that a set of gambles $\mathcal{D}$ is *maximal* if it is coherent and there does not exist any $X \notin \mathcal{D}$ such that $\overline{\mathcal{D} \cup \{X\}}$ is coherent.

Lemma

If $\mathcal{D}$ is coherent and $-X \notin \mathcal{D}, \quad X \neq 0$, then $\overline{\mathcal{D} \cup \{X\}}$ is coherent.

Theorem

A coherent set of gambles $\mathcal{D}$ is maximal if and only if $X \in \mathcal{D}$ xor $-X \in \mathcal{D}$, for all $X \in \mathcal{L}, \quad X \neq 0$.

Lemma

Let $\mathcal{D}$ be a maximal set of gambles and let $\underline{P}$ and $\overline{P}$ be respectively the lower and the upper previsions associated to it. Then $\underline{P}(B) = \overline{P}(B), \ \forall B \subseteq \Omega$.

Definition

If we have a sequence of nested sets $\Omega = C_0 \supset C_1 \supset \cdots \supset C_n = \emptyset$, and $B \subseteq \Omega$, then the layer of B with respect to this sequence, will be the minimum value of i such that $B \cap (C_i \setminus C_{i+1}) \neq \emptyset$. It will be denoted by $\mathrm{layer}(B)$.

Theorem

If $\mathcal{D}$ is maximal then there is a sequence of nested sets $\Omega = C_0 \supset C_1 \supset \cdots \supset C_n = \emptyset$ and a sequence of probability measures $P_0, \ldots, P_{n-1}$ satisfying the following conditions:

1. *for each probability P_i, $P_i(C_i \setminus C_{i+1}) = 1$, $P_i(\omega) > 0$ for any $\omega \in C_i \setminus C_{i+1}$,*
2. *for each $A \subseteq B \subseteq \Omega$, if $i = \mathrm{layer}(B)$, then $\underline{P}(A|B) = \overline{P}(A|B) = P_i(A|B)$, where $\underline{P}(A|B)$ and $\overline{P}(A|B)$ are the lower and upper probabilities computed from $\mathcal{D}_B$.*

Coletti and Scozzafava (2002)

Maximal Gambles

Theorem

There exists at least one maximal set of gambles containing a coherent set.

Theorem

If $\mathcal{D}$ is coherent, then $\mathcal{D} = \cap_{i \in I} \mathcal{D}_i$, where $\mathcal{D}_i$ are maximal coherent gambles containing $\mathcal{D}$.

Correspondence (Sequences of probabilities <—> Maximal coherent sets) non one-to-one

$\Omega = \{\omega_1, \omega_2\}$ and $P_0(\omega_1) = P_0(\omega_2) = 0.5$.
Any gamble with $X(\omega_1) + X(\omega_2) > 0$ is desirable.
Given $Y(\omega_1) = 1, Y(\omega_2) = -1$.
We can have Y desirable xor $-Y$ desirable.

Alternative model one-to-one:

D1". If $X \in \mathcal{D}$, then there is $\epsilon > 0$, such that $-X + \epsilon\,\mathrm{supp}(X) \notin \mathcal{D}$.

More Work

- More general representation schemes?
- Algorithms for them?
- Local computation
- Independence and local computation