Exchangeability for sets of desirable gambles

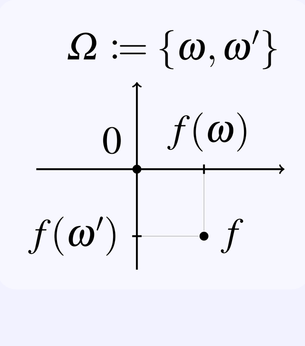
Desirability

General context: experiments & gambles

A finite possibility space Ω of outcomes of some experiment.

A subject who is uncertain about the experiment's outcome.

Gambles $f \in \mathscr{G}(\Omega) := \mathbb{R}^\Omega$, interpreted as uncertain rewards: $f(\omega)$ when the experiment's outcome is ω.



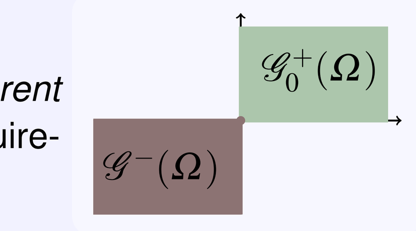
A gamble f is *desirable* to the subject if he accepts the following transaction:
(i) the actual outcome ω is determined, and
(ii) the subject's capital is changed by $f(\omega)$.

The zero gamble 0 is not desirable.

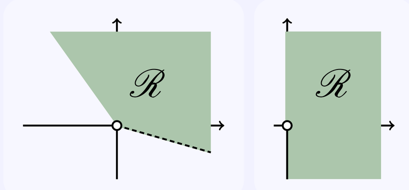
Coherent sets of desirable gambles

A subject's set of desirable gambles $\mathscr{R} \subseteq \mathscr{G}(\Omega)$ models his beliefs about the experiment's outcome.

The set of desirable gambles $\mathscr{R}$ is *coherent* if it satisfies the following rationality requirements: $(f, f_1, f_2 \in \mathscr{G}(\Omega),\ \lambda > 0)$



D1. if $f = 0$ then $f \notin \mathscr{R}$;
D2. if $f > 0$ then $f \in \mathscr{R}$ [accepting partial gain];
D3. if $f \in \mathscr{R}$ then $\lambda f \in \mathscr{R}$ [scaling];
D4. if $f_1, f_2 \in \mathscr{R}$ then $f_1 + f_2 \in \mathscr{R}$ [combination].



Requirements D3 and D4 make $\mathscr{R}$ a *cone*: $\operatorname{coni}(\mathscr{R}) = \mathscr{R}$.

Sets of weakly desirable gambles

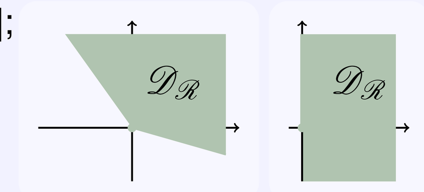
The subject considers a gamble f in $\mathscr{G}(\Omega)$ *weakly desirable* if by adding any desirable gamble to it, another desirable gamble is obtained; so if $f' \in \mathscr{R}$ then $f + f' \in \mathscr{R}$.

The subject's set of weakly desirable gambles is

$$\mathscr{D}_\mathscr{R} := \{f \in \mathscr{G}(\Omega) \colon f + \mathscr{R} \subseteq \mathscr{R}\}.$$

The set of weakly desirable gambles $\mathscr{D}_\mathscr{R}$ corresponding to a coherent set of desirable gambles $\mathscr{R}$ satisfies the following properties: $(f, f_1, f_2 \in \mathscr{G}(\Omega),\ \lambda \geq 0)$

WD1. if $f < 0$ then $f \notin \mathscr{D}_\mathscr{R}$ [avoiding partial loss];
WD2. if $f \geq 0$ then $f \in \mathscr{D}_\mathscr{R}$ [accepting partial gain];
WD3. if $f \in \mathscr{D}_\mathscr{R}$ then $\lambda f \in \mathscr{D}_\mathscr{R}$ [scaling];
WD4. if $f_1, f_2 \in \mathscr{D}_\mathscr{R}$ then $f_1 + f_2 \in \mathscr{D}_\mathscr{R}$ [combination].

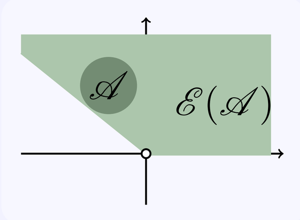


$\mathscr{D}_\mathscr{R}$ is the closure of $\mathscr{R}$, excluding gambles in $\mathscr{G}_0^-(\Omega)$.

Assessments & their natural extension

An assessment can consist of a set $\mathscr{A} \subseteq \mathscr{G}(\Omega)$ considered desirable by the subject.

The assessment $\mathscr{A}$ *avoids non-positivity* if the intersection of $\operatorname{coni}(\mathscr{A})$ and $\mathscr{G}_0^-(\Omega)$ is empty.



The *natural extension* of $\mathscr{A}$ is

$$\mathscr{E}(\mathscr{A}) := \operatorname{coni}\big(\mathscr{G}_0^+(\Omega) \cup \mathscr{A}\big).$$

If $\mathscr{A}$ avoids non-positivity, then $\mathscr{E}(\mathscr{A})$ is the smallest coherent set of desirable gambles including $\mathscr{A}$.

Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event B of Ω.

Contingent on observing B, the subject models his beliefs using an *updated* set of desirable gambles, the subset of $\mathscr{G}(B)$ given by

$$\mathscr{R}\rfloor B := \{f_B \colon I_B f \in \mathscr{R}\}.$$

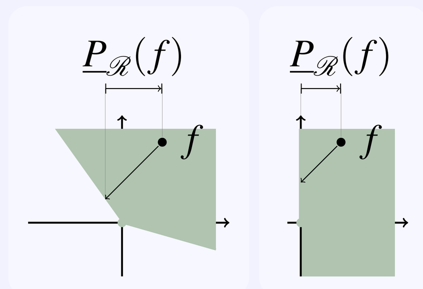
If $\mathscr{R}$ is a coherent set of desirable gambles on Ω, then $\mathscr{R}\rfloor B$ is a coherent set of desirable gambles on B.

Coherent previsions & desirability

The *lower prevision* of a gamble f associated to a set of desirable gambles $\mathscr{A}$ is

$$\underline{P}_\mathscr{A}(f) := \sup\{\mu \in \mathbb{R} \colon f - \mu \in \mathscr{A}\}.$$

Its conjugate upper prevision $\overline{P}_\mathscr{A}(f)$ is equal to $-\underline{P}_\mathscr{A}(-f)$.



A lower prevision $\underline{P}$ is coherent if there exists some coherent set of desirable gambles $\mathscr{R}$ such that $\underline{P} = \underline{P}_\mathscr{R} = \underline{P}_{\mathscr{D}_\mathscr{R}}$.

Coherent lower previsions are less expressive uncertainty models than coherent sets of desirable gambles.

Exchangeability

Specific context: finite sequences

The experiment consists of the observation of the value of a sequence $X_1, \dots, X_N$ of random variables for which $\mathscr{X}$ is the finite set of possible values. So the possibility space Ω is $\mathscr{X}^N$ and $x = (x_1, \dots, x_N)$ is one of its elements.

$\mathscr{X} := \{●, ●\},\ N := 3$

$x := (●, ●, ●)$

$\mathscr{P}_N$ is the set of all permutations π of the index set $\{1, \dots, N\}$.

The associated permutation of $\mathscr{X}^N$ is defined by $(\pi x)_k = x_{\pi(k)}$.

It is lifted to a permutation π^t of $\mathscr{G}(\mathscr{X}^N)$ by letting $\pi^t f = f \circ \pi$.

With every sequence of observations corresponds a *count vector* in

$$\mathscr{N}^N = \{m \in \mathbb{N}^\mathscr{X} \colon \textstyle\sum_{z \in \mathscr{X}} m_z = N\}.$$

The *counting map* $T^N \colon \mathscr{X}^N \to \mathscr{N}^N$ maps a sequence x to a vector $m = T^N(x)$.

$T^3(●, ●, ●) = (1, 2)$

Permuted sequences have the same count vector; a *permutation invariant atom* is

$$[m] := \{y \in \mathscr{X}^N \colon T^N(y) = m\}.$$

$$[1, 2] = \{(●, ●, ●), (●, ●, ●), (●, ●, ●)\}$$

Exchangeability

If a subject assesses that $X_1, \dots, X_N$ are *exchangeable*, this means that for any gamble f and any permutation π, he finds exchanging $\pi^t f$ for f weakly desirable, because he is indifferent between them.

The negation invariant space of all such exchange gambles is

$$\mathscr{D}_{\mathscr{P}_N} := \{f - \pi^t f \colon f \in \mathscr{G}(\mathscr{X}^N) \text{ and } \pi \in \mathscr{P}_N\}.$$

If $\mathscr{D}_{\mathscr{P}_N}$ consists of weakly desirable gambles, then so does its conical hull $\mathscr{D}_{\mathscr{U}_N} = \operatorname{coni}(\mathscr{D}_{\mathscr{P}_N}) = \operatorname{span}(\mathscr{D}_{\mathscr{P}_N})$.

A coherent set $\mathscr{R}$ of desirable gambles on $\mathscr{X}^N$ is called *exchangeable* if $\mathscr{D}_{\mathscr{U}_N} \subseteq \mathscr{D}_\mathscr{R}$, or equivalently, if

$$\mathscr{D}_{\mathscr{U}_N} + \mathscr{R} \subseteq \mathscr{R}.$$

If $\mathscr{R}$ is coherent and exchangeable then it is also *permutable*: for all f in $\mathscr{R}$ and all π in $\mathscr{P}_N$, it holds that $\pi^t f \in \mathscr{R}$.

Exchangeable natural extension

The assessment $\mathscr{A}$ *avoids non-positivity under exchangeability* if $\mathscr{A} + \mathscr{D}_{\mathscr{U}_N}$ avoids non-positivity.

The *exchangeable natural extension* of $\mathscr{A}$ is

$$\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A}) := \mathscr{D}_{\mathscr{U}_N} + \mathscr{E}(\mathscr{A}).$$

If $\mathscr{A}$ avoids non-positivity under exchangeability, then $\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A})$ is the smallest exchangeable coherent set of desirable gambles including $\mathscr{A}$.

Updating exchangeable models

The subject observes the values $\check{x} = (\check{x}_1, \check{x}_2, \dots, \check{x}_{\check{n}})$ or the count vector $\check{m}$ in $\mathscr{N}^{\check{n}}$ of the first $\check{n}$ variables $X_1, \dots, X_{\check{n}}$; this means observing the event $\{\check{x}\} \times \mathscr{X}^{\hat{n}}$ or $[\check{m}] \times \mathscr{X}^{\hat{n}}$. We are interested in inferences about the remaining $\hat{n} = N - \check{n}$ variables.

Contingent on observing $\check{x}$ or $\check{m}$, the subject models his beliefs using updated sets of desirable gambles, the subsets of $\mathscr{G}(\mathscr{X}^{\hat{n}})$ that are

$$\mathscr{R}\rfloor\check{x} := \{f(\check{x}, \cdot) \colon I_{\{\check{x}\} \times \mathscr{X}^{\hat{n}}} f \in \mathscr{R}\},$$
$$\mathscr{R}\rfloor\check{m} := \{f(\check{y}, \cdot) \colon I_{[\check{m}] \times \mathscr{X}^{\hat{n}}} f \in \mathscr{R} \text{ and } \check{y} \in [\check{m}]\}.$$

If $\mathscr{R}$ is a coherent and exchangeable set of desirable gambles on $\mathscr{X}^N$, then $\mathscr{R}\rfloor\check{x}$ and $\mathscr{R}\rfloor\check{m}$ are coherent and exchangeable sets of desirable gambles on $\mathscr{X}^{\hat{n}}$.

Under exchangeability, count vectors are *sufficient statistics*: if $T^{\check{n}}(\check{x}) = \check{m}$, then $\mathscr{R}\rfloor\check{x} = \mathscr{R}\rfloor\check{m}$.

Exchangeable previsions

A lower prevision $\underline{P}$ on $\mathscr{G}(\mathscr{X}^N)$ is *exchangeable* if there is some exchangeable coherent set of desirable gambles $\mathscr{R}$ such that $\underline{P} = \underline{P}_\mathscr{R}$.

Representation

Moving between sequence gambles and count gambles

$$\begin{tikzcd}
\mathscr{G}(\mathscr{X}^N) \arrow[rr, "\mathrm{ex}^N"] \arrow[dr, "\mathrm{MuHy}^N"'] & & \mathscr{G}_{\mathscr{P}_N}(\mathscr{X}^N) \\
& \mathscr{G}(\mathscr{N}^N) \arrow[ur, "\mathrm{T}^N"'] &
\end{tikzcd}$$

The set of permutation invariant sequence gambles is

$$\mathscr{G}_{\mathscr{P}_N}(\mathscr{X}^N) := \{f \in \mathscr{G}(\mathscr{X}^N) \colon (\forall \pi \in \mathscr{P}_N)\pi^t f = f\}.$$

The projection of a sequence gamble f onto a permutation invariant sequence gamble is

$$\mathrm{ex}^N(f) := \tfrac{1}{N!}\textstyle\sum_{\pi \in \mathscr{P}_N} \pi^t f = \sum_{m \in \mathscr{N}^N} \mathrm{MuHy}^N(f|m) I_{[m]},$$

where its value on an invariant atom $[m]$ is given by

$$\mathrm{MuHy}^N(f|m) := \tfrac{1}{|[m]|}\textstyle\sum_{y \in [m]} f(y).$$

The count gamble corresponding to the sequence gamble f is

$$\mathrm{MuHy}^N(f) := \mathrm{MuHy}^N(f|\cdot).$$

The permutation invariant sequence gamble in a one-to-one correspondence with the count gamble g is

$$\mathrm{T}^N(g) := g \circ T^N.$$

Representation

A set of desirable gambles $\mathscr{R}$ on $\mathscr{X}^N$ is coherent and exchangeable iff there is some coherent set $\mathscr{S}$ of desirable gambles on $\mathscr{N}^N$ – its *count representation* – such that

$$\mathscr{R} = (\mathrm{MuHy}^N)^{-1}(\mathscr{S}),$$

and in that case this $\mathscr{S}$ is uniquely determined by

$$\mathscr{S} = \{g \in \mathscr{G}(\mathscr{N}^N) \colon \mathrm{T}^N(g) \in \mathscr{R}\} = \mathrm{MuHy}^N(\mathscr{R}).$$

Exchangeable natural extension & representation

The assessment $\mathscr{A} \subseteq \mathscr{X}^N$ *avoids non-positivity under exchangeability* if $\mathrm{MuHy}^N(\mathscr{A})$ avoids non-positivity.

A nice result: $\mathrm{MuHy}^N(\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A})) = \mathscr{E}(\mathrm{MuHy}^N(\mathscr{A}))$.

$$\begin{tikzcd}
2^{\mathscr{G}(\mathscr{X}^N)} \arrow[r, "\mathrm{MuHy}^N"] \arrow[d, "\mathscr{E}_{\mathrm{ex}}^N"] & 2^{\mathscr{G}(\mathscr{N}^N)} \arrow[d, "\mathscr{E}"] \\
2^{\mathscr{G}(\mathscr{X}^N)} \arrow[r, "\mathrm{MuHy}^N"] & 2^{\mathscr{G}(\mathscr{N}^N)}
\end{tikzcd}$$

Representing updated models

The subject observes the values $\check{x} = (\check{x}_1, \check{x}_2, \dots, \check{x}_{\check{n}})$ or the count vector $\check{m} = T^{\check{n}}(\check{x})$ in $\mathscr{N}^{\check{n}}$ of the first $\check{n}$ variables $X_1, \dots, X_{\check{n}}$.

If $\mathscr{R}$ is a coherent and exchangeable set of desirable gambles on $\mathscr{X}^N$, then the representation of the two – because of sufficiency – identical updated models he uses is

$$\mathscr{S}\rfloor\check{m} := \mathrm{MuHy}^{\hat{n}}(\mathscr{R}\rfloor\check{m}).$$

This representation is *not* an updated model of the representation $\mathscr{S} = \mathrm{MuHy}^N(\mathscr{R})$ of $\mathscr{R}$. They are however related by

$$\mathscr{S}\rfloor m = \{g(\check{m} + \cdot) \colon L_{\check{m}} g \in \mathscr{S}\},$$

where we use the *likelihood function*, defined for every count vector m in $\mathscr{N}^n$ by

$$L_{\check{m}}(m) := \frac{|[\check{m}]|\,|[m - \check{m}]|}{|[m]|},$$

which is zero when $m \ngeq \check{m}$.

Exchangeable previsions & representation

A lower prevision $\underline{P}$ on $\mathscr{G}(\mathscr{X}^N)$ is coherent and exchangeable iff there is some coherent lower prevision $\underline{Q}$ on $\mathscr{G}(\mathscr{N}^N)$ – its *count representation* – such that $\underline{P} = \underline{Q} \circ \mathrm{MuHy}^N$. In that case $\underline{Q}$ is uniquely determined by $\underline{Q} = \underline{P} \circ \mathrm{T}^N$.

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