On the use of a new discrepancy measure to correct incoherent assessments and to aggregate conflicting opinions based on imprecise conditional probabilities

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1. BASIC NOTIONS

We formalize the domain of the evaluation through a finite family of conditional events of the type

$$\mathcal{E} = [E_1|H_1, \ldots, E_n|H_n]$$

and the sample space spanned by the basic events $E_1, \ldots, E_n, H_1, \ldots, H_n$ is given by

$$\Omega = \{\omega_1, \ldots, \omega_k\},$$

where ω_j represents a generic atom, in some context named “possible world ”.

The numerical part of the assessment is elicited through interval values

$$\mathbf{lub} = ([lb_1, ub_1], \ldots, [lb_n, ub_n])$$

thought as honest ranges for the probabilities $p_i = P(E_i|H_i)$, $i = 1, \ldots, n$.

Denoting by $\mathcal{M}$ the set of coherent precise conditional assessments compatible with $(\mathcal{E}, \mathbf{lub})$, i.e.

$$\mathcal{M} := \{P \text{ coherent} \, | lb_i \leq P(E_i|H_i) \leq ub_i, i = 1, \ldots, n\}$$

we shall focus on the situations with an empty $\mathcal{M}$ that characterize incoherent assessments (*with uniform loss*).

Every probability distribution $\alpha : \mathcal{P}(\Omega) \to \mathbb{R}$ corresponds to a non-negative vector $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_k]$, with $\alpha_j = \alpha(\omega_j)$; then for every event E it will be $\alpha(E) = \sum_{\omega_j \subseteq E} \alpha_j$.

We will refer to a nested hierarchy of probability distributions over Ω:

- $\mathcal{A} := \{\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_k] \,|\, \sum_1^k \alpha_i = 1, \alpha_j \geq 0, \quad j = 1, \ldots, k\}$ is the whole set of probability distributions on Ω;

- $\mathcal{A}_0 := \left\{\boldsymbol{\alpha} \in \mathcal{A} | \alpha(H^0) = \alpha(\bigvee H_i) = 1\right\}$ is the subset of probability distributions on Ω that concentrate all the probability mass on the contemplated scenarios;

- $\mathcal{A}_1 := \{\boldsymbol{\alpha} \in \mathcal{A}_0 | \alpha(H_i) > 0, i = 1, \ldots, n\}$ is the subset of p.d. on Ω that give positive probability to every scenario;

- $\mathcal{A}_2$ is the subset of probability distributions that avoid boundary values $\{0, 1\}$ for the conditional probabilities.

Associated to any precise assessment $\mathbf{p} = [p_1, \ldots, p_n] \in (0,1)^n$ over $\mathcal{E}$ we can introduce a scoring rule

$$S(\mathbf{p}) := \sum_{i=1}^{n} |E_i H_i| \ln p_i + \sum_{i=1}^{n} |\neg E_i H_i| \ln(1 - p_i)$$

where $|\cdot|$ is the indicator function of unconditional events and the value of $S(\mathbf{p})$ when the atom ω_j occurs is

$$S_j(\mathbf{p}) = \sum_{i:\, E_i H_i \supset \omega_j} \ln p_i + \sum_{i:\, \neg E_i H_i \supset \omega_j} \ln(1 - p_i).$$

We can now introduce the "discrepancy" between a precise assessment $\mathbf{p}$ over $\mathcal{E}$ and a distribution $\boldsymbol{\alpha} \in \mathcal{A}_2$,

with respect to its induced conditional coherent assessment $\mathbf{q}_{\boldsymbol{\alpha}}$, as

$$\Delta(\mathbf{p}, \boldsymbol{\alpha}) := E_{\boldsymbol{\alpha}}(S(\mathbf{q}_{\boldsymbol{\alpha}}) - S(\mathbf{p})) = \sum_{j=1}^{k} \alpha_j [S_j(\mathbf{q}_{\boldsymbol{\alpha}}) - S_j(\mathbf{p})]\,.$$

It is possible to extend by continuity the previous definition of $\Delta(\mathbf{p}, \boldsymbol{\alpha})$ to any distribution $\boldsymbol{\alpha}$ in $\mathcal{A}_0$ with

$$\Delta(\mathbf{p}, \boldsymbol{\alpha}) = \sum_{i=1}^{n} \ln(\frac{q_i}{p_i}) \alpha(E_i H_i) + \ln(\frac{1 - q_i}{1 - p_i}) \alpha(\neg E_i H_i) = \sum_{i=1}^{n} \alpha(H_i) \left(q_i \ln(\frac{q_i}{p_i}) + (1 - q_i) \ln(\frac{1 - q_i}{1 - p_i}) \right).$$

For the *discrepancy measure* $\Delta(\mathbf{p}, \boldsymbol{\alpha})$ the following properties hold:

- $\Delta(\mathbf{p}, \boldsymbol{\alpha}) \geq 0 \quad \forall \boldsymbol{\alpha} \in \mathcal{A}$;
- $\Delta(\mathbf{p}, \boldsymbol{\alpha}) = 0$ iff $\mathbf{p} \equiv \mathbf{q}_{\boldsymbol{\alpha}}$;
- $\Delta(\mathbf{p}, \cdot)$ is convex on $\mathcal{A}_2$;
- $\Delta(\mathbf{p}, \cdot)$ always admits a minimum on $\mathcal{A}_0$;
- If $\Delta(\mathbf{p}, \cdot)$ attains its minimum on $\mathcal{A}_1$ there is a unique coherent assessment $\mathbf{q}_{\underline{\alpha}}$ on $\mathcal{E}$ such that $\Delta(\mathbf{p}, \underline{\alpha})$ is minimum;
- If $\Delta(\mathbf{p}, \cdot)$ attains its minimum value on $\mathcal{A}_0 \setminus \mathcal{A}_1$, then any distribution $\boldsymbol{\alpha} \in \mathcal{A}_0$ that minimizes $\Delta(\mathbf{p}, \cdot)$ induces the same significant conditional probabilities $(\mathbf{q}_\alpha)_j$ on the conditional events $E_j | H_j$ such that $\alpha(H_j) > 0$.

3. CORRECTING INCOHERENT ASSESSMENTS

By fixing an index $f \in \{1,\dots,n\}$, we can find two coherent assessments $\underline{\mathbf{q}}_f$ and $\overline{\mathbf{q}}_f$ on $\mathcal{E}$, induced by the solutions of the following two parametric optimization problems:

$$\text{minimize } \Delta(\mathbf{v},\boldsymbol{\alpha}) \tag{1}$$

under the constraints

$$\begin{gathered} v_f = lb_f \quad \text{or} \quad v_f = ub_f \\ \forall i \neq f \quad lb_i \leq v_i \leq ub_i, i \in \{1,\dots,n\} \\ \sum_{j:\,\omega_j \subset E_k H_k} \alpha_j = q_k \sum_{j:\,\omega_j \subset H_k} \alpha_j, k = 1,\dots,n \\ \boldsymbol{\alpha} \in \mathcal{A}_0. \end{gathered}$$

By letting the index f vary over the full range $1,\dots,n$ we obtain a set of $2n$ coherent assessments

$$\mathcal{Q} = \{\underline{\mathbf{q}}_f, \overline{\mathbf{q}}_f, f = 1,\dots,n\}.$$

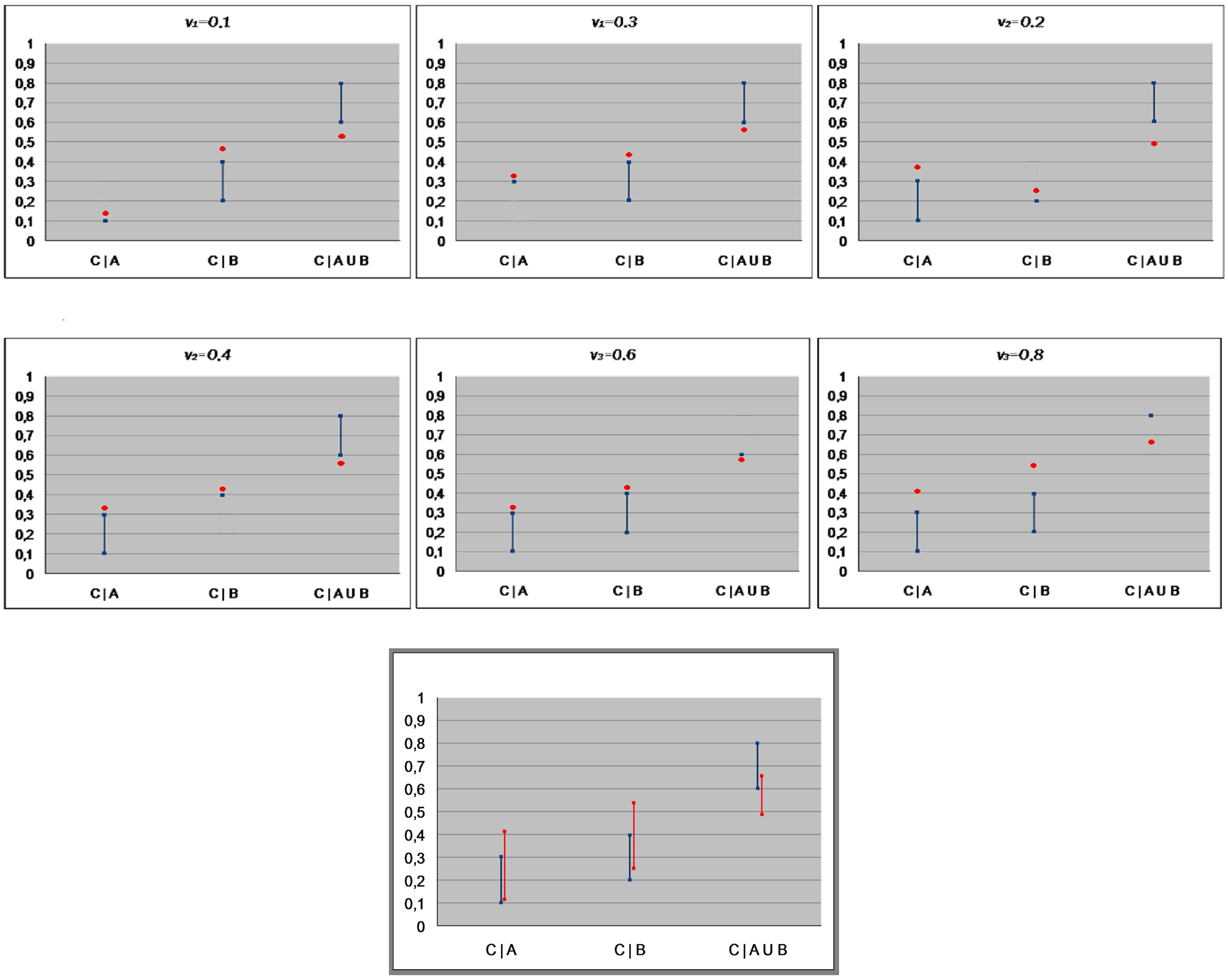
Hence the imprecise assessment on $\mathcal{E}$

$$lc_i := \min_{\tilde{\mathbf{q}} \in \mathcal{Q}} \tilde{\mathbf{q}}(E_i|H_i) \qquad uc_i := \max_{\tilde{\mathbf{q}} \in \mathcal{Q}} \tilde{\mathbf{q}}(E_i|H_i),$$

is coherent and can be adopted as correction of $\mathbf{lub}$.

Example 1

$\mathcal{E}$	$C\|A$	$C\|B$	$C\|A \vee B$
lb_i	0.1	0.2	0.6
ub_i	0.3	0.4	0.8



Evaluations are assessed on $\mathcal{E}^s = [E_{1.s}|H_{1.s}, \ldots, E_{n.s}|H_{n.s}]$, with the index $s \in S$ expressing the different sources; $\mathcal{E} = \bigvee_{s \in S} \mathcal{E}^s$ is the joint domain.

If we have two different ranges $[lb'_i, ub'_i]$ and $[lb''_i, ub''_i]$ associated to the same $E_i|H_i \in \mathcal{E}$, we can associate the second interval to a new conditional event $E''_i|H''_i$ and increase the logical relationships with

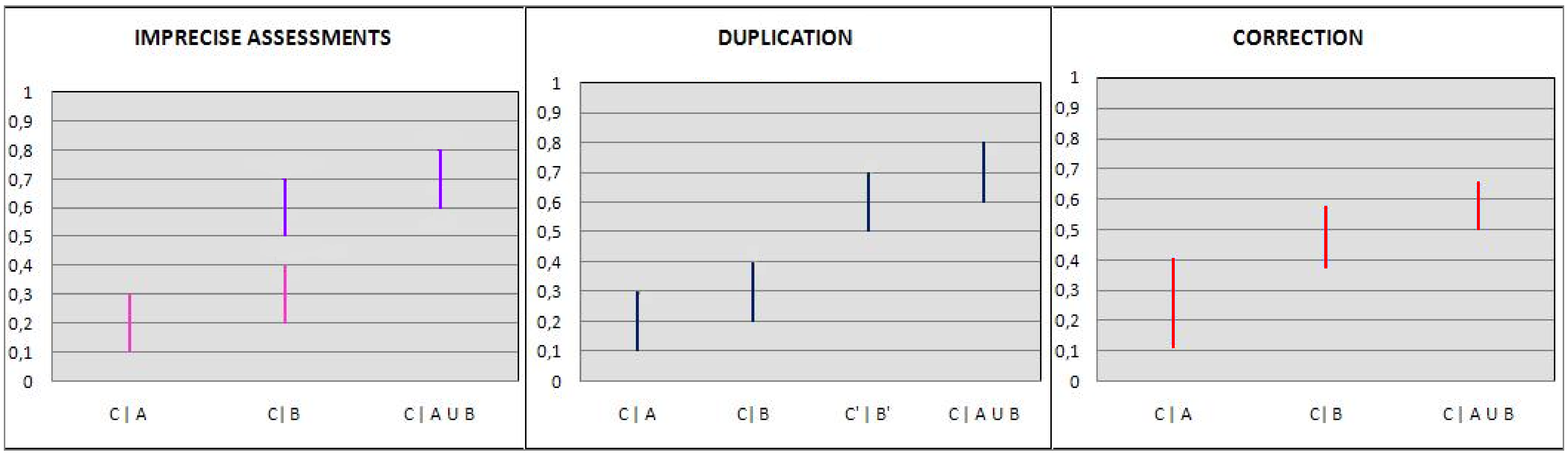
$$E_i H_i \equiv E''_i H''_i \quad , \quad H_i \equiv H''_i .$$

In this way, we will have the different opinions joined in a single imprecise (incoherent) assessment $(\mathcal{E}, \mathbf{lub})$; and its correction $(\mathcal{E}, \mathbf{luc})$ will represent an aggregation result.

Since $\mathbf{luc}$ is a coherent imprecise assessment, equal intervals will be associated to coincident elements of $\mathcal{E}$.

Example 2

	$C\|A$	$C\|B$	$C\|A \vee B$
$\mathbf{lub}'$	$[.1, .3]$	$[.2, .4]$	$-$
$\mathbf{lub}''$	$-$	$[.5, .7]$	$[.6, .8]$



5. WEIGHTED AGGREGATION

It is possible to associate different weights to the elements of the joined assessment $(\mathcal{E}, \mathbf{lub})$;

denoting by $\mathbf{w} = [w_1, \ldots, w_n]$ such weights, the expression of $\Delta(\mathbf{v}, \boldsymbol{\alpha})$ in the optimization problems (1) becomes

$$\Delta^{\mathbf{w}}(\mathbf{v}, \boldsymbol{\alpha}) := \sum_{i=1}^{n} w_i \alpha(Hi) \left(q_i \ln(\frac{q_i}{v_i}) + (1 - q_i) \ln \frac{(1 - q_i)}{(1 - v_i)} \right) \tag{2}$$

Example 3

	$C\|A$	$C\|B$	$C\|A \vee B$
$\mathbf{lub}'$	$[.1, .3]$	$[.2, .4]$	$[.6, .8]$
$\mathbf{lub}''$	$[.1, .3]$	$[.5, .7]$	$[.6, .8]$

$\rightsquigarrow$

$\mathcal{E}$	$C\|A$	$C\|B$	$C''\|B''$	$C\|A \vee B$
$\mathbf{lub}$	$[.1, .3]$	$[.2, .4]$	$[.5, .7]$	$[.6, .8]$
$\mathbf{w}$	2	1	1	2