Multivariate Models and Confidence Intervals: A Local Random Set Approach

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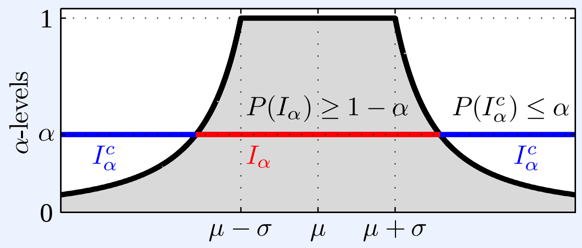
The univariate case

Non-parametric models, Tchebycheff

We model the variability of an uncertain parameter X with $\mu = \mathsf{E}(X)$ and $\sigma^2 = \mathsf{V}(X)$ as sole information by a nested family $\mathbf{I} = \{I_\alpha\}_{\alpha\in(0,1]}$ of non-parametric confidence intervals (cf. Dubois2004, Oberguggenberger2007)

$$\boxed{I_\alpha = \left[\mu - \tfrac{\sigma}{\sqrt{\alpha}}, \mu + \tfrac{\sigma}{\sqrt{\alpha}}\right], \quad \alpha \in (0,1]}$$

using Tchebycheff's inequality $P(|X-\mu| > \frac{\sigma}{\sqrt{\alpha}}) \le \alpha$. Then $P(I_\alpha^c) \le \alpha$ and $P(I_\alpha) = 1 - P(I_\alpha^c) \ge 1-\alpha$.



Local upper probability $\overline{P}_\alpha$ at level α

Equipping the two intervals I_α and I_α^c with weights

$$m(I_\alpha) = P(I_\alpha) \quad\text{and}\quad m(I_\alpha^c) = P(I_\alpha^c)$$

we get a local random set where $m(I_\alpha)$ corresponds to the confidence we have in the interval I_α.

The local upper probability $\overline{P}_\alpha(A)$ at level α for an event A is

$$\boxed{\overline{P}_\alpha(A) = m(I_\alpha)\,\chi(A\cap I_\alpha \ne \varnothing) + m(I_\alpha^c)\,\chi(A\cap I_\alpha^c \ne \varnothing).}$$

(χ indicator function)

Three different cases for A:		$\overline{P}_\alpha(A) \in$
(i)	$A\cap I_\alpha = \varnothing$	$[0,\alpha]$
(ii)	$A\cap I_\alpha^c = \varnothing$	$[1-\alpha, 1]$
(iii)	$A\cap I_\alpha \ne \varnothing$ and $A\cap I_\alpha^c \ne \varnothing$	1

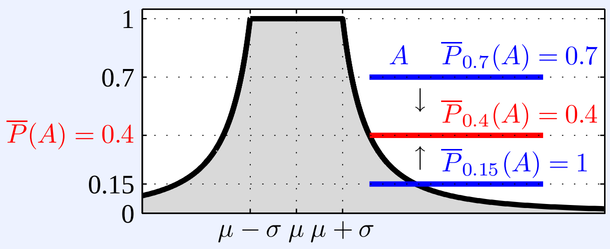
If A has the role of the "bad and undesired" event, case (i) is the most interesting one, because its meaning is:

> *If A is outside the confidence interval I_α at confidence level $1-\alpha$, then we can say for sure that A occurs only with probability α, at most.*

To avoid interval-valued upper probabilities we take always the upper bounds of $\overline{P}_\alpha$ in the above table.

Upper probability $\overline{P}(A)$

$$\boxed{\overline{P}(A) = \inf_{\alpha\in(0,1]} \overline{P}_\alpha(A) = \inf\{\alpha\in(0,1] : I_\alpha \cap A = \varnothing\} = \alpha^*}$$



Interpretation of I as random set and fuzzy set

Together with the uniform distribution on the interval $(0,1]$, the family $\mathbf{I}$ of confidence intervals is a random set where the upper probability $\overline{P}(A)$ (Plausibility) is

$$\overline{P}(A) = \mathrm{Pl}(A) = \int_{\beta:\, I_\beta\cap A\ne\varnothing} \mathrm{d}\beta = 1 - \int_{\beta:\, I_\beta\cap A=\varnothing} \mathrm{d}\beta = 1 - \int_{\inf\{\beta:\, I_\beta\cap A=\varnothing\}=\alpha^*}^{1} \mathrm{d}\beta = \alpha^*.$$

$\mathbf{I}$ is a fuzzy number defined by the nested α-level sets I_α and membership function μ given by the endpoints of the I_α. The upper probability $\overline{P}(A)$ (Possibility) is

$$\begin{aligned}\overline{P}(A) = \mathrm{Pos}(A) &= \sup\{\mu(x) :\ x\in A\} = \\ &= \sup\{\alpha\in(0,1] :\ I_\alpha \cap A \ne \varnothing\} = \\ &= \inf\{\alpha\in(0,1] :\ I_\alpha\cap A = \varnothing\} = \alpha^*.\end{aligned}$$

The results of all three approaches coincide!

The multivariate case

Goal: Formula similar to the univariate version

Given: Families $\mathbf{I}_1,\dots,\mathbf{I}_n$ of confidence intervals.

We have several possibilities of choice for the set of confidence intervals considered to be combined and for the weights used for the local joint random set.

Combination of marginal confidence intervals

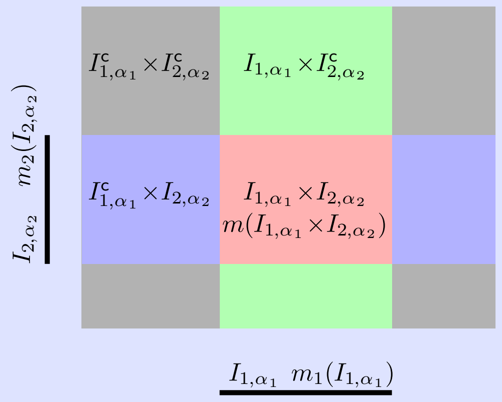
$\mathbf{J} = \{J_{\boldsymbol\alpha}\}_{\boldsymbol\alpha\in S}$ is the family of all joint confidence sets

$$J_{\boldsymbol\alpha} = I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n}, \quad I_{k,\alpha_k}\in\mathbf{I}_k$$

with $\boldsymbol\alpha = (\alpha_1,\dots,\alpha_n)$ depending on which set S of indices $\boldsymbol\alpha$ is considered:

- Random set independence like: $S = S_{\mathrm R} = (0,1]^n$.
- Fuzzy set independence like: $S = S_{\mathrm F} = \{\boldsymbol\alpha\in(0,1]^n : \alpha_1=\alpha_2=\dots=\alpha_n\} \subseteq S_{\mathrm R}$.

Local joint focal sets, 2-dimensional



Local upper probability $\overline{P}_{\boldsymbol\alpha}(A)$

In the following an event A in the expression $\overline{P}_{\boldsymbol\alpha}(A)$ is always satisfying $A\cap I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n} = \varnothing$.

We do not care about how an event A hits the remaining joint focals sets. Assuming the worst case (hitting all remaining focals sets) relieves us from dealing with unbounded focal sets.

This means

$$\boxed{\begin{aligned}\overline{P}_{\boldsymbol\alpha}(A) &= \overline{P}((I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n})^c)\\ &= 1 - m(I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n}).\end{aligned}}$$

The local joint weight $m(I_{1,\alpha_1}\times I_{2,\alpha_2})$, 2-dim.

The weights of all four joint focal sets has to be chosen in a way that the horizontal and vertical sums in the following table lead to the marginal weights m_i:

$m_2(I_{2,\alpha_2}^c) = \alpha_2$	$m(I_{1,\alpha_1}\times I_{2,\alpha_2}^c)$	$m(I_{1,\alpha_1}^c\times I_{2,\alpha_2}^c)$
$m_2(I_{2,\alpha_2}) = 1-\alpha_2$	$m(I_{1,\alpha_1}\times I_{2,\alpha_2})$	$m(I_{1,\alpha_1}^c\times I_{2,\alpha_2})$
	$m_1(I_{1,\alpha_1}) = 1-\alpha_1$	$m_1(I_{1,\alpha_1}^c) = \alpha_1$

$m(I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n})$, $\overline{P}_{\boldsymbol\alpha}$, random set independence

$$m(I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n}) = \prod_{i=1}^n m_i(I_{i,\alpha_i}) = \prod_{i=1}^n (1-\alpha_i)$$

$$\boxed{\overline{P}_{\boldsymbol\alpha}(A) = 1 - \prod_{i=1}^n (1-\alpha_i)}$$

Used if the uncertain variables are independent.

$m(I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n})$, $\overline{P}_{\boldsymbol\alpha}$, lower/upper bounds

Using the lower and upper bounds of Fréchet for joint probability distributions we get for the joint weight

$$\max\Big(\sum_{i=1}^n m(I_{i,\alpha_i}) - n + 1, 0\Big) \le \le m(I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n}) \le \min_{i=1,\dots,n} m(I_{i,\alpha_i}).$$

$m(I_{i,\alpha_i}) = 1-\alpha_i$ and $\overline{P}_{\boldsymbol\alpha}(A) = 1 - m(I_{1,\alpha_1}\times\dots\times I_{n,\alpha_n})$ lead to bounds for the local upper probability:

$$\boxed{\max_{i=1,\dots,n}(\alpha_i) \le \overline{P}_{\boldsymbol\alpha}(A) \le \min(\alpha_1+\dots+\alpha_n, 1).}$$

Used if nothing is known about interactions between the uncertain variables.

Levels of the joint confidence set

The three approaches have an influence on the level

$$\ell(\boldsymbol\alpha) = \begin{cases}\max\limits_{i=1,\dots,n}(\alpha_i) & \text{lower bound,}\\ 1-\prod_{i=1}^n(1-\alpha_i) & \text{random set independence,}\\ \min(\alpha_1+\dots+\alpha_n, 1) & \text{upper bound}\end{cases}$$

of the joint confidence sets, but not on the sets itself.

- Upper probability similar to the univariate case:

$$\boxed{\overline{P}_\ell^S(A) = \inf_{\boldsymbol\alpha\in S}\{\ell(\boldsymbol\alpha) : J_{\boldsymbol\alpha}\cap A = \varnothing\}.}$$

- Upper distribution function: $\overline{F}_\ell^S(x) = \overline{P}_\ell^S((\infty, x])$.

Notations depending on ℓ and S

All possible combinations of confidence intervals, $S = S_{\mathrm R}$ → superscript R:

Notation	level $\ell(\boldsymbol\alpha)$	
$\overline{P}^{\mathrm R}_{\mathrm{lower}}$	$\max\limits_{i=1,\dots,n}(\alpha_i)$	lower Fréchet bound
$\overline{P}^{\mathrm R}_{\mathrm{indep}}$	$1-\prod_{i=1}^n(1-\alpha_i)$	random set independence
$\overline{P}^{\mathrm R}_{\mathrm{upper}}$	$\min\left(\sum_{i=1}^n \alpha_i, 1\right)$	upper Fréchet bound

Combinations of confidence intervals of the same level α only, $S = S_{\mathrm F}$ → superscript F:

Notation	level $\ell(\alpha)$	
$\overline{P}^{\mathrm F}_{\mathrm{lower}}$	α	lower Fréchet bound
$\overline{P}^{\mathrm F}_{\mathrm{indep}}$	$1-(1-\alpha)^n$	random set independence
$\overline{P}^{\mathrm F}_{\mathrm{upper}}$	$\min(n\alpha, 1)$	upper Fréchet bound

For comparison with classical aproaches:

Notation	
$\overline{P}_{\mathrm R}$	random set independence
$\overline{P}_{\mathrm F}$	fuzzy set independence
$\overline{P}_{\mathrm U}$	unknown interaction

The ordering of the upper probabilities

$\overline{P}_{\mathrm F}(A) = \overline{P}^{\mathrm R}_{\mathrm{lower}}(A) = \overline{P}^{\mathrm F}_{\mathrm{lower}}(A)$

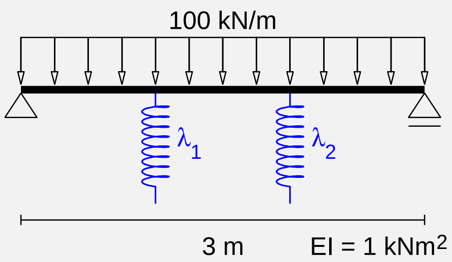
$\overline{P}_{\mathrm R}(A) \le \overline{P}^{\mathrm R}_{\mathrm{indep}}(A) \le \overline{P}^{\mathrm F}_{\mathrm{indep}}(A)$

$\overline{P}_{\mathrm U}(A) \le \overline{P}^{\mathrm R}_{\mathrm{upper}}(A) \le \overline{P}^{\mathrm F}_{\mathrm{upper}}(A)$

Numerical example

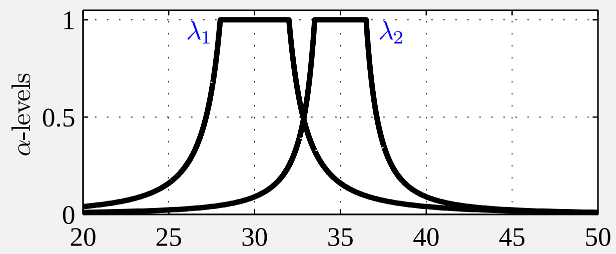
Beam bedded on two springs

The spring constants λ_1 and λ_2 are uncertain.



Modelling the uncertainty of λ_1 and λ_2

variable	expectation	variance
λ_1; λ_2	30; 35	2; 1.5



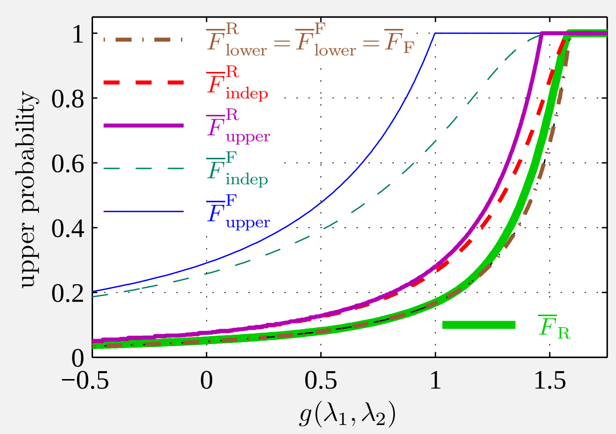
Criterion of failure: $g(\lambda_1,\lambda_2) \le 0$

Failure function

$$g(\lambda_1,\lambda_2) = M_{\mathsf{yield}} - \max_{x\in[0,3]} |M(x,\lambda_1,\lambda_2)|$$

where $M(x,\lambda_1,\lambda_2)$ is the bending moment at point $x\in[0,3]$ depending on the two spring constants λ_1,λ_2 and $M_{\mathsf{yield}} = 12$ kNm the elastic limit moment.

Upper probability distributions $\overline{F}_\ell^S(g(\lambda_1,\lambda_2))$



The upper probabilities of failure are the results at 0.