Characterizing Factuality in Normal Form Sequential Decision Making

Nathan Huntley and Matthias Troffaes

Durham University, Department of Mathematical Sciences

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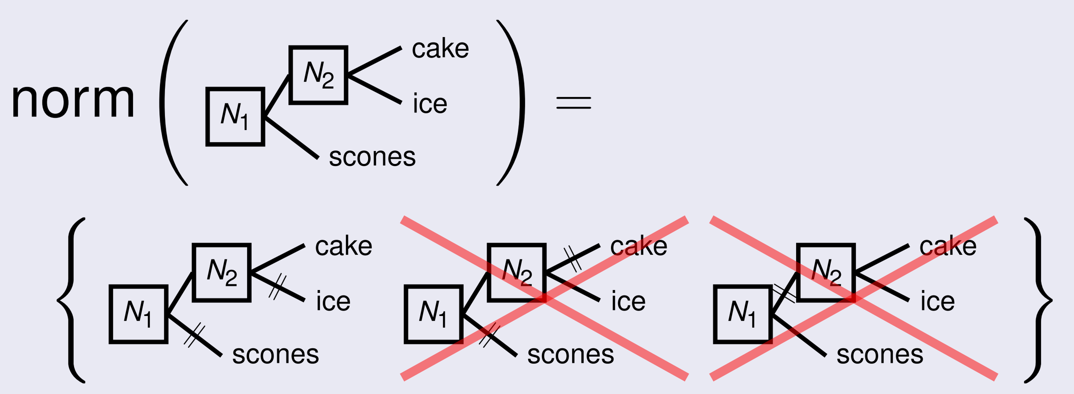
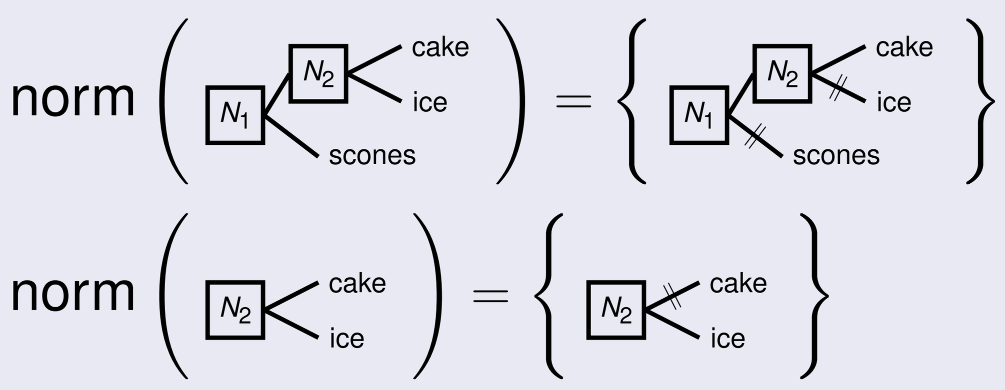
Background

What is Normal Form?



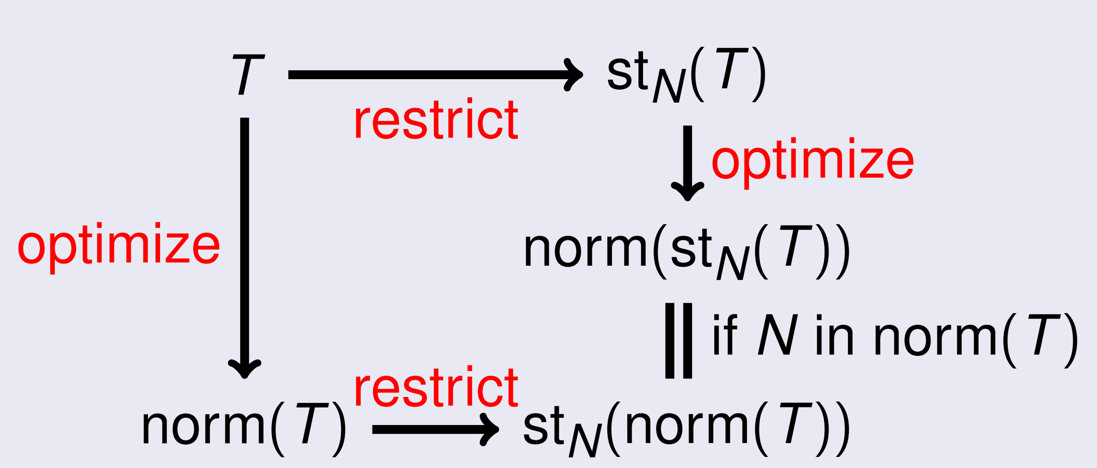
- A normal form **decision** of a decision tree T describes a subject's choices in T in all eventualities.
- A normal form **operator** norm maps every decision tree T to a set of normal form decisions of T.

What is Counterfactuality?



This subject is counterfactual as his choice between cake and ice cream depends on the tree in which the choice is embedded.

Factuality can be represented by a commuting diagram:



Choice Functions

- A normal form decision induces a gamble, which maps each outcome to a reward.
- A choice function opt maps any set $\mathcal{X}$ of gambles, conditional on an event A, to an optimal subset:

$$\emptyset \neq \mathrm{opt}(\mathcal{X}|A) \subseteq \mathcal{X}.$$

Gambles are convenient for defining a normal form operator. To compare normal form decisions, compare their gambles:

$$\mathrm{norm}_{\mathrm{opt}}(T) = \{ \text{normal form decision } U \text{ of } T: \mathrm{gamb}(U) \subseteq \mathrm{opt}(\mathrm{gamb}(T)|\,\mathrm{ev}(T))\}.$$

Main Result

Factuality Theorem

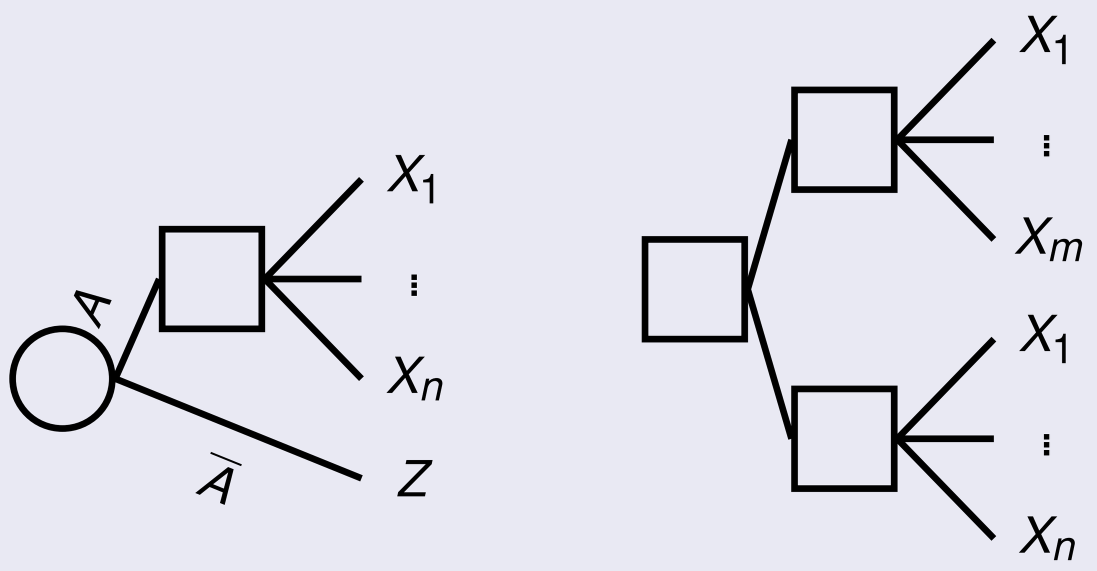
A normal form operator induced by a choice function is factual if and only if opt satisfies these three properties:

- Conditioning property. If $\{X, Y\} \subseteq \mathcal{X}$ and $AX = AY$, then
$$X \in \mathrm{opt}(\mathcal{X}|A) \iff Y \in \mathrm{opt}(\mathcal{X}|A).$$
- Intersection property. If $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathrm{opt}(\mathcal{X}|A) \cap \mathcal{Y} \neq \emptyset$, then
$$\mathrm{opt}(\mathcal{Y}|A) = \mathrm{opt}(\mathcal{X}|A) \cap \mathcal{Y}.$$
- Mixture property.
$$\mathrm{opt}(A\mathcal{X} \oplus \overline{A}Z|B) = A\,\mathrm{opt}(\mathcal{X}|A \cap B) \oplus \overline{A}Z.$$

Note: in the above, we have omitted some technical details.

Necessity

Necessity of the three properties can be observed from these two simple trees.



Sufficiency

See forthcoming paper [1] (summary of proof in conference paper).

References

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Consequences

Total Preorder Theorem

The intersection property is equivalent to:

- Total preorder property. For every event $A \neq \emptyset$, there is a total preorder $\succeq_A$ on gambles such that
$$\mathrm{opt}(\mathcal{X}|A) = \max_{\succeq_A}(\mathcal{X})$$

Hence, any choice function that is *not* induced by a total preorder induces a counterfactual normal form operator.

Backward Induction Theorem

Backward induction solves a decision tree by recursively applying $\mathrm{norm}_{\mathrm{opt}}$ from right to left, so gambles that are (hopefully!) non-optimal can be removed early on: call this normal form operator $\mathrm{back}_{\mathrm{opt}}$.

If $\mathrm{norm}_{\mathrm{opt}}$ is factual, then $\mathrm{norm}_{\mathrm{opt}} = \mathrm{back}_{\mathrm{opt}}$ (but not the other way around!).

“No Imprecision” Theorem

	Property			
	C	I	M	BI
E-admissibility	✓		✓	✓
Maximality	✓		✓	✓
Γ-maximin	✓	✓		
Interval Dominance	✓			

We are not aware of any choice functions, other than maximizing expected utility, that induce factual normal form operators.

Discussion & Conclusions

- Factuality imposes strong restrictions.
- All imprecise probability choice functions, that we know of, violate intersection or mixture.
- Factuality provides a compelling argument **against** imprecision (or at least, against incomplete orderings).
- Factual normal form operators other than those induced by choice functions are possible, but often have unwelcome properties [2].
- Factual extensive form solutions are easier to find [3].