Nonparametric Predictive Multiple Comparisons with Censored Data and Competing Risks

Tahani Maturi, Pauline Coolen-Schrijner, Frank Coolen

Dept of Mathematical Sciences, Durham University, Durham DH1 3LE, UK

Nonparametric Predictive Inference

Nonparametric predictive inference (NPI) is based on Hill's assumption $A_{(n)}$ (Hill, 1968), which implies direct (lower and upper) probabilities for a future observable random quantity, based on observed values of n related random quantities (Coolen, 2006). Suppose that $X_1, \ldots, X_n, X_{n+1}$ are positive, continuous and exchangeable random quantities representing lifetimes. Let their ordered observed values be denoted by $x_1 < x_2 < \ldots < x_n$, and let $x_0 = 0$ and $x_{n+1} = \infty$ for ease of notation. For positive X_{n+1}, representing a future observation, based on n observations, $A_{(n)}$ assigns $P(X_{n+1} \in (x_i, x_{i+1})) = 1/(n+1)$ for $i = 0, 1, \ldots, n$.

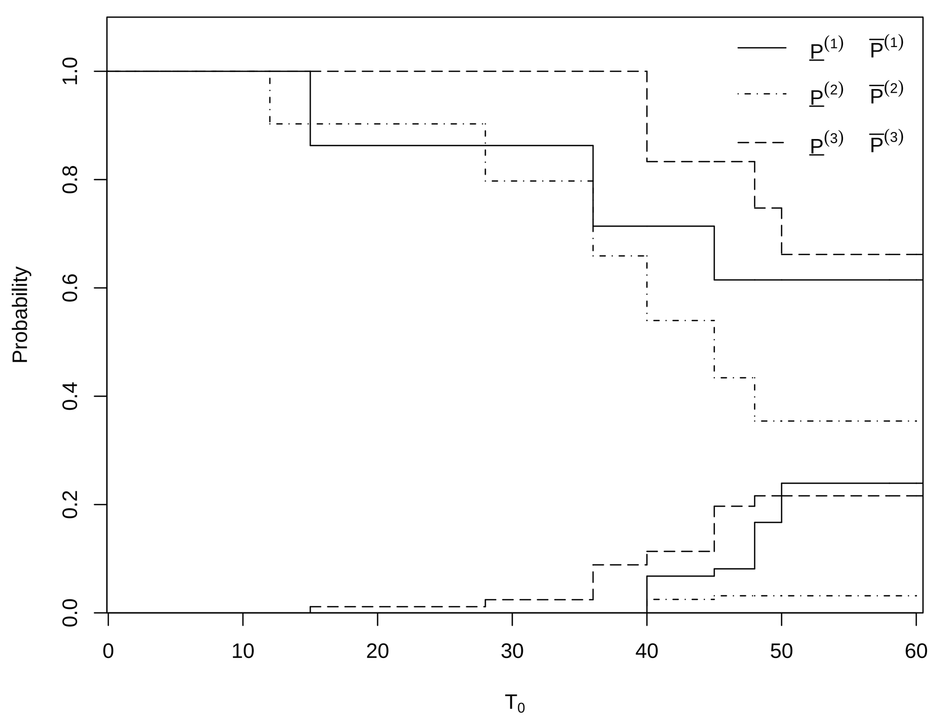
Coolen and Yan (2004) presented rc-$A_{(n)}$ as a generalization of $A_{(n)}$ for right-censored data, using the extra assumption that, at a moment of censoring, the residual time-to-failure of a right-censored unit is exchangeable with the residual time-to-failure of all other units that have not yet failed or been censored. Coolen and Yan (2003) developed NPI for comparison of two groups of lifetime data including right-censored observations. By applying the appropriate rc-$A_{(n)}$ assumption for each group, their method is based on comparing the next observation from each group. However, they did not consider situations with more than two groups, nor the effect of early termination of the lifetime experiment or the specific features of progressive censoring and competing risks.

Early termination of experiment

In order to save costs or time, an experiment to compare lifetimes of units in different groups may be terminated before all units have failed. We assume that all units are placed simultaneously on a lifetime experiment which is terminated at a certain specified time, which may also be the moment a specified number of failures have occurred. This is known as *precedence testing* in the literature (Balakrishnan & Ng, 2006). We presented NPI for comparison of two groups of lifetime data with early termination of the experiment. Then we extend this for several groups, with a variety of multiple comparisons goals including, selecting the best group, the subset of best groups and the subset that consists the best group. Recently we present further generalized results, by developing NPI for comparison of multiple groups of lifetime data including right-censored observations, and with possible early termination of the experiment.

Example: Early termination

Desu and Raghavarao (2004) present recorded times (months) until promotion at a large company, for 19 employees in $k = 3$ departments. The data are: Dept 1: $15, 20^+, 36, 45, 58, 60$ ($n_1 = 6$); Dept 2: $12, 25^+, 28, 30^+, 30^+, 36, 40, 45, 48$ ($n_2 = 9$); Dept 3: $30^+, 40, 48, 50$ ($n_3 = 4$), where $+$ indicates that the employee left the company at that length of service before getting promotion (i.e. right-censored). We consider at which department the data suggest that one needs to work the longest to get a promotion. This data contains tied observations, in NPI these are dealt with by assuming that they differ by a very small amount, in such a way that the lower (or upper) probability of interest is smallest (largest) over all possible ways to break the ties.



To illustrate NPI, we assume that all these employees started working at this company at the same time, and that one considers the data after T_0 months, so all larger observations in the data above are treated as being right-censored at T_0. For several values of T_0, the lower and upper probabilities, $\underline{P}^{(l)}$ and $\overline{P}^{(l)}$, for the event that one has to work the longest in Dept l ($l = 1, 2, 3$), are visualized above. There is no value of T_0 for which the data would strongly indicate that one of the departments leads to longest time to promotion, according to the criteria that $\overline{P}^{(j)} < \underline{P}^{(l)}\ \forall j \neq l$. For several T_0, for example $T_0 = 17$, both the lower and upper probabilities for Dept 3 are greater than the lower and upper probabilities, respectively, for Dept 1 and for Dept 2. So this provides a weak indication that Dept 3 leads to the longest times until promotion.

Progressive censoring

In progressive censoring, some units are randomly removed from the experiment at several stages. For NPI for a progressive Type-II censoring scheme with $R = (R_1, \ldots, R_r)$, where R_i is the number of units that are removed from the experiment at the ith failure. We presented the appropriate rc-$A_{(n)}$ assumption for a nonnegative random quantity X_{n+1} on the basis of data including r real and $n - r$ progressively censored observations. For comparing two independent groups, X and Y, for which n_x and n_y units, respectively, are placed on a lifetime experiment. Both groups are progressively Type-II censored with the schemes $R^x = (R^x_1, \ldots, R^x_{r_x})$ and $R^y = (R^y_1, \ldots, R^y_{r_y})$. We derived the NPI lower and the upper probabilities that the next observation from group Y is greater than the next observation from group X. We also introduced NPI comparisons in case of progressive Type-I and Type-II progressively hybrid censoring.

Competing risks

In competing risks, a unit is subject to failure from one of k distinct failure modes. Such failure observations are obtained for n units, and that failure modes causing failures are known and independent. For the NPI approach, let the failure time of a future item be denoted by X_{n+1}, and let $X_{j,n+1}$ be the failure time including indication of the actual failure mode j. Then the data per failure mode consist of a number of observed failure times for failures caused by the specific failure mode considered, and right-censoring times for failures caused by other failure modes. Hence we can apply rc-$A_{(n)}$ per failure mode j, for inference on $X_{j,n+1}$. We derived the NPI lower and upper probabilities for the random quantity representing the failure time of the next unit, with all k failure modes considered, i.e. $X_{n+1} = \min\limits_{1 \leq j \leq k} X_{j,n+1}$.

Example: Competing risks

The data contain information about 36 units of a new model of a small electrical appliance, where the lifetime observation per unit consists of the number of completed cycles of use until the unit failed (Lawless, 2003). In the study, there were 18 failure modes (FM) in which an appliance could fail.

# cycles	FM	# cycles	FM	# cycles	FM	# cycles	FM
11	1	1167	9	2551	9	3112	9
35	15	1594	2	2565	-	3214	9
49	15	1925	9	2568	9	3478	9
170	6	1990	9	2702	10	3504	9
329	6	2223	9	2761	6	4329	9
381	6	2327	6	2831	2	6367	-
708	6	2400	9	3034	9	6976	9
958	10	2451	5	3034	9	7846	9
1062	5	2471	9	3059	6	13403	-

The most frequently occurring failure mode in these data is FM9, which caused 17 units to fail. We consider how likely it is that the next unit, say unit 37, would fail due to FM9, assuming it would undergo the same test and its number of completed cycles would be exchangeable with these numbers for the 36 units reported. Let us group all failure modes other than FM9 together, and consider these jointly as a failure mode, so we consider the NPI approach with 2 failure modes, FM9 and, say, 'other failure mode' (OFM). There are still three units that do not fail (i.e. right-censored), indicated by '-'.

The NPI lower and upper probabilities for the event that unit 37 will fail due to FM9 are

$$\underline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.4358$$

$$\overline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.5804$$

Now consider the situation with FM9 and FM6 separately, and by grouping all the other failure modes together into OFM. Then the corresponding lower and upper probabilities for the event that unit 37 will fail due to FM9, FM6 or OFM, are

$$\underline{P}\left(X_{37}^{FM9} < \min\left\{X_{37}^{FM6}, X_{37}^{OFM}\right\}\right) = 0.3915$$

$$\overline{P}\left(X_{37}^{FM9} < \min\left\{X_{37}^{FM6}, X_{37}^{OFM}\right\}\right) = 0.5804$$

$$\underline{P}\left(X_{37}^{FM6} < \min\left\{X_{37}^{FM9}, X_{37}^{OFM}\right\}\right) = 0.1749$$

$$\overline{P}\left(X_{37}^{FM6} < \min\left\{X_{37}^{FM9}, X_{37}^{OFM}\right\}\right) = 0.3279$$

$$\underline{P}\left(X_{37}^{OFM} < \min\left\{X_{37}^{FM6}, X_{37}^{FM9}\right\}\right) = 0.2265$$

$$\overline{P}\left(X_{37}^{OFM} < \min\left\{X_{37}^{FM6}, X_{37}^{FM9}\right\}\right) = 0.3808$$

References

Balakrishnan, N. and Ng, H.K.T. (2006). *Precedence-Type Tests and Applications*. Wiley.

Coolen, F.P.A. (2006). On nonparametric predictive inference and objective Bayesianism. *Journal of Logic, Language and Information*, 15, 21-47.

Coolen, F.P.A. and Yan, K.J. (2003). Nonparametric predictive comparison of two groups of lifetime data. *ISIPTA'03*, pp. 148–161.

Coolen, F.P.A. and Yan, K.J. (2004). Nonparametric predictive inference with right-censored data. *Journal of Statistical Planning and Inference*, 126, 25–54.

Desu, M. M. and Raghavarao, D. (2004). *Nonparametric Statistical Methods for Complete and Censored Data*. Chapman & Hall.

Hill, B.M. (1968): Posterior distribution of percentiles: Bayes theorem for sampling from a finite population. *Journal of the American Statistical Association*, *63*, 677–691.

Lawless, J.F. (2003). *Statistical models and methods for lifetime data*. Wiley.