Natural extension as a limit of regular extensions

Enrique Miranda (U. of Oviedo, Spain) and Marco Zaffalon (IDSIA, Switzerland)

Abstract

What is *coherence* of lower previsions? And their *natural extension*? These questions have a very clear answer when we work with desirable gambles but not that much from the dual viewpoint of probability.
We show that both coherence and the natural extension can be regarded as related to the existence of a sequence of unconditional credal sets from which, by Bayes' rule, the original assessments can recovered as well as all their natural extensions.
Furthermore, we discuss the difference between the natural extension, and what we call the weak natural extension (i.e., that based on weak coherence). We argue that most approaches in the literature compute weak natural extensions, which we show are not enough informative compared to natural extensions.
Our results are valid for finite spaces and conditional lower previsions with non-linear domains.

Introduction

Tools: Variables $X_1,\dots,X_n$ taking finitely many values, and coherent lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$.

Preminary results: We give new characterisations of avoiding uniform sure and partial loss based on the existence of dominating conditional linear previsions.

Weak natural extension: How can we extend weakly coherent lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ to new ones? Result: the (so-called weak natural) extension can be made through conditioning the smallest unconditional prevision $\underline{P}(X_1,\dots,X_n)$ that is weakly coherent with them.

Main result, (strong) coherence: $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ are jointly coherent if and only if there is a sequence of unconditional lower previsions $\underline{P}_\epsilon(X_1,\dots,X_n)$, $\epsilon\in\mathbb{R}^+$, s.t. by applying Bayes' rule whenever possible to the mass functions in the set equivalent to $\underline{P}_\epsilon(X_1,\dots,X_n)$, we recover the original conditional lower previsions in the limit.

Main result, natural extension: the natural extension of the original assessments to a new lower prevision $\underline{P}_{m+1}(X_{O_{m+1}}|X_{I_{m+1}})$ is nothing else but the application of Bayes' rule to $\underline{P}_\epsilon(X_1,\dots,X_n)$ with $\epsilon\to 0$.

Zero probabilities: We relate the need of the sequence $\underline{P}_\epsilon(X_1,\dots,X_n)$, $\epsilon\in\mathbb{R}^+$, to the existence of events with zero lower probability and show that in this case the weak natural extension can be much less informative than the natural extension.

Credits: Walley, Pelessoni and Vicig have introduced these ideas while restricting the attention to events (rather than gambles) and therefore to finitely many probabilistic assessments. Our work builds upon those ideas, while generalising them so that the only actual restriction now is the finiteness of the spaces.

Basic notions

Variables: $X_1,\dots,X_n$, taking values in respective *finite* sets $\mathcal{X}_1,\dots,\mathcal{X}_n$.
The vector of variables $(X_j)_{j\in J}$ is denoted by X_J.
The space of possibilities is $\mathcal{X}^n := \times_{j\in\{1,\dots,n\}}\mathcal{X}_j$.

Conditional lower prevision (CLP): $\underline{P}(X_O|X_I)$, with domain $\mathcal{H}\subseteq\mathcal{K}$, where $\mathcal{K}$ is the set of all gambles that depend on X_O, X_I, represents a subject's beliefs about the gambles that depend on the outcome of the variables $\{X_j, j\in O\}$, after coming to know the outcome of the variables $\{X_j, j\in I\}$.
We *always* take $\underline{P}(X_O|X_I)$ to be *separately coherent*.

Basic model: A collection of CLPs, i.e., $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$.

Initial results + weak coherence

Avoiding uniform sure loss: $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ avoid uniform sure loss iff there are dominating weakly coherent conditional linear previsions with (full) domains $\mathcal{K}^1,\dots,\mathcal{K}^m$.

Avoiding partial loss: $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ avoid partial loss iff there are dominating coherent conditional linear previsions with domains $\mathcal{K}^1,\dots,\mathcal{K}^m$.

Weak natural extension: Let $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ be separately coherent conditional lower previsions with domains $\mathcal{H}^1,\dots,\mathcal{H}^m$. The following are equivalent:

(WC1) They are weakly coherent.

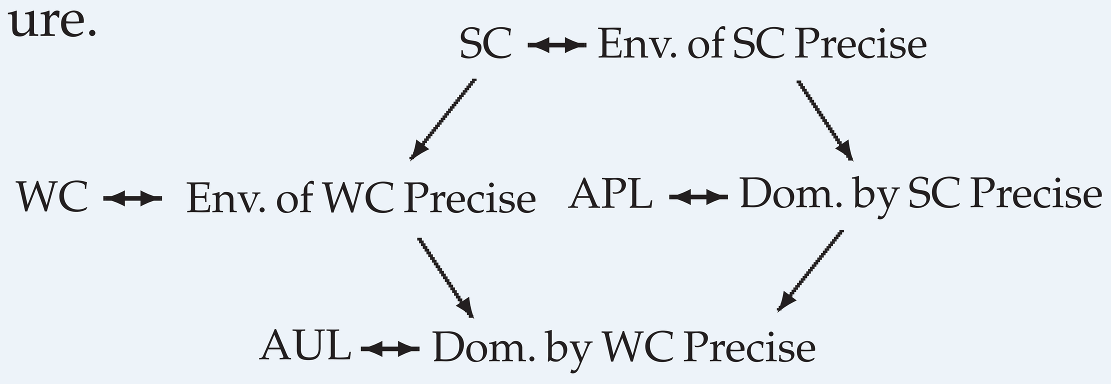
(WC2) They are the lower envelopes of a class of weakly coherent conditional linear previsions, $\{P_1^\lambda(X_{O_1}|X_{I_1}),\dots,P_m^\lambda(X_{O_m}|X_{I_m}) : \lambda\in\Lambda\}$.

(WC3) There is a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}^n)$ which is weakly coherent with them.

(WC4) There is a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}^n)$ which is pairwise coherent with them.

Moreover, we give an explicit formula for the smallest coherent lower prevision $\underline{P}$ in (WC3) and (WC4).

We summarise the relationships between the different consistency conditions when all the referential spaces are finite in the following figure.



Keys: SC = strongly coherent; WC = weakly coherent; AUL = avoiding uniform sure loss; APL = avoiding partial loss; Env. = envelope; Dom. = dominated.

Weak natural extension

Weak natural extension: The smallest CLP $\underline{P}_{m+1}(X_{O_{m+1}}|X_{I_{m+1}})$ with domain $\mathcal{K}^{m+1}$ which is weakly coherent with $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ is given, for every $f\in\mathcal{K}^{m+1}, z_{m+1}\in\mathcal{X}_{I_{m+1}}$, by $\underline{P}_{m+1}(f|z_{m+1}) :=$

$$\begin{cases}\min_{x\in\pi^{-1}_{I_{m+1}}(z_{m+1})} f(x) \text{ if } \underline{P}(z_{m+1})=0\\ \min\{P(f|z_{m+1}) : P\geq\underline{P}\} \text{ otherwise,}\end{cases}$$

where $\underline{P}$ is the smallest one in (WC3) or (WC4).
The weak natural extension can be too little informative.

Example: Consider X_1, X_2 taking values in $\mathcal{X}:=\{1,2\}$. Define coherent *linear* previsions $P(X_1), P(X_2|X_1)$ using the assessments $P(X_1=1):=1, P(X_2=1|X_1=1):=0.5, P(X_2=1|X_1=2):=1$. We obtain a unique coherent linear joint by total probability. It assigns probability zero to $X_1=2$. As a consequence, the weak natural extension of $P(X_1), P(X_2|X_1)$ is *vacuous* for X_2 conditional on $X_1=2$. On the other hand, it follows from the coherence of $P(X_1), P(X_2|X_1)$ that the natural extension yields back the original linear prevision $P(X_2|X_1=2)$, which tells us that $X_2=1$ with certainty given $X_1=2$.

Main results, strong coherence

$\mathcal{M}(\epsilon)$: For every $\epsilon>0$, let $\mathcal{M}(\epsilon)$ be the set of unconditional linear previsions satisfying

$$P(f_j|z_j)\geq\underline{P}_j(f_j|z_j)-\epsilon R(f_j)$$

whenever $P(z_j)>0$, and for every $f_j\in\mathcal{H}^j, z_j\in\mathcal{X}_{I_j}, j=1,\dots,m$, where $R(f_j):=\max f_j-\min f_j$.

Approximating conditionals: For every $\epsilon>0$, define the approximating conditionals $\underline{R}^\epsilon_{m+1}(f|z_{m+1}) := \inf\{P(f|z_{m+1}) : P\in\mathcal{M}(\epsilon), P(z_{m+1})>0\}$ and their limits $\underline{F}_{m+1}(X_{O_{m+1}}|X_{I_{m+1}}) := \lim_{\epsilon\to 0}\underline{R}^\epsilon_{m+1}(X_{O_{m+1}}|X_{I_{m+1}})$.

Main result: $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ weakly coherent and avoiding partial loss. Then the natural extension $\underline{E}_{m+1}(X_{O_{m+1}}|X_{I_{m+1}})$ coincides with $\underline{F}_{m+1}(X_{O_{m+1}}|X_{I_{m+1}})$.

Characterising coherence: Let $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ be separately coherent CLPs. They are coherent if and only if they are the pointwise limits of a sequence of coherent CLPs defined by regular extension.

Essential references

- E. Miranda and G. de Cooman, Coherence and independence in non-linear spaces, T.R., 2005.
- P. Walley, *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, 1991.
- P. Walley, R. Pelessoni, and P. Vicig, Direct algorithms for checking consistecy and making inferences for conditional probability assessments, *JSPI*, 126:119–151, 2004.