# The role of generalized p-boxes in imprecise probability theories


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We have recently introduced generalized p-boxes [1], encompassing possibility distributions and classical p-boxes. Due to their simplicity and interpretation, they are promising models. Here, we study their computational aspects for various information processings, to evaluate their potential as practical uncertainty models.


## Definitions

Variable assuming values on $X = \{x_1, \dots, x_N\}$. Two mappings are said to be **comonotonic** if there is a common permutation $\sigma$ of $\{1, 2, \dots, N\}$ such that $f(x_{\sigma(1)}) \geq f(x_{\sigma(2)}) \geq \dots \geq f(x_{\sigma(N)})$ and $f'(x_{\sigma(1)}) \geq f'(x_{\sigma(2)}) \geq \dots \geq f'(x_{\sigma(N)})$.

**Definition 1 (Gen. P-box)** *A generalised p-box $[\underline{F}, \overline{F}]$ is a pair of comonotonic mappings $\underline{F}, \overline{F}$, $\underline{F} : \mathcal{X} \to [0,1]$ and $\overline{F} : \mathcal{X} \to [0,1]$ from $\mathcal{X}$ to $[0,1]$ such that $\underline{F} \leq \overline{F}$ and there is at least one element $x$ in $\mathcal{X}$ for which $\overline{F}(x) = \underline{F}(x) = 1$, these bounds ensuring that $[\underline{F}, \overline{F}]$ will be a coherent lower probability.*

Induce weak order on $\mathcal{X}$ such that $x \leq_{[\underline{F},\overline{F}]} y$ iff $\overline{F}(x) \leq \overline{F}(y)$ and $\underline{F}(x) \leq \underline{F}(y)$. Elements $x_1, \dots, x_N$ are indexed such that $i < j \to \overline{F}(x_i) \leq \overline{F}(x_j)$ and $\underline{F}(x_i) \leq \underline{F}(x_j)$.

**Definition 2 ($[\underline{F}, \overline{F}]$-connected subsets)** *Subset $C \subseteq \mathcal{X}$ is called $[\underline{F}, \overline{F}]$-connected if it can be expressed as a union of consecutive elements $x_k$, that is*

$$C = \{x_k \in \mathcal{X} | 0 < i \leq k \leq j \leq N\}$$

**Definition 3 ($\sqsubseteq_{[\underline{F},\overline{F}]}$-ordering)** *$A = \{x_i, \dots, x_j\}, B = \{x_{i'}, \dots, x_{j'}\} \subseteq \mathcal{X}$ two $[\underline{F}, \overline{F}]$-connected sets. $\sqsubseteq_{[\underline{F},\overline{F}]}$-ordering between them defined as*

$$A \sqsubseteq_{[\underline{F},\overline{F}]} B \to \begin{cases} i \leq i' \\ j \leq j'. \end{cases}$$

## Links with other models

A gen. p-box $[\underline{F}, \overline{F}]$ generates the following models:

- **Probability sets:** $\mathcal{P}_{[\underline{F},\overline{F}]} = \{P | \underline{F}(x_i) \leq P(\{x_1, \dots, x_i\}) \leq \overline{F}(x_i)\}$.
  → classical p-boxes retrieved when $\mathcal{X} = \mathbb{R}$ and sets $A_i = (-\infty, x_i]$

- **Random sets:** RS mapping $m : \wp(\mathcal{X}) \to [0,1]$ s.t. $\sum_{E \in \wp(\mathcal{X})} m(E) = 1$. Associate to $A \subseteq X$ a lower meas. $Bel(A) = \sum_{E, E \subseteq A} m(E)$ and a set $\mathcal{P}_m = \{P | \forall A, P(A) \geq Bel(A)\}$.

Denote by $0 = \gamma_0 < \gamma_1 < \dots < \gamma_M = 1$ the $M$ distinct values taken by $\overline{F}, \underline{F}$, then $[\underline{F}, \overline{F}]$ equivalent to the following random set, for $j = 1, \dots, M$

$$\begin{cases} E_j = \{x_i \in X | (\overline{F}(x_i) \geq \gamma_j) \wedge (\underline{F}(x_{i-1}) < \gamma_j)\}, \\ m(E_j) = \gamma_j - \gamma_{j-1}. \end{cases} \quad (1)$$

All $E_j$ are $[\underline{F}, \overline{F}]$-connected and form a complete order w.r.t. $[\underline{F}, \overline{F}]$-ordering.

- **Possibility distributions:** mapping $\pi : X \to [0,1]$ generating, for $A \subseteq X$ an upper measure $\Pi(A) = \sup_{x \in A} \pi(x)$ and an associated set $\mathcal{P}_\pi = \{P | \forall A, P(A) \leq \Pi(A)\}$.

Gen. p-box modelled by pair of distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$ s.t. for $i = 1, \dots, N$,

$$\pi_{\overline{F}}(x_i) = \overline{F}(x_i) \quad \text{and} \quad \pi_{\underline{F}}(x_i) = 1 - \underline{F}(x_{i-1})\}, \quad (2)$$

in the sense that $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{\pi_{\underline{F}}} \cap \mathcal{P}_{\pi_{\overline{F}}}$.

→ the pair $[\pi_{\overline{F}}, 1 - \pi_{\underline{F}}]$ forms a **cloud** [3]

## 1. Computing probability bounds

Lower probability of a $[\underline{F}, \overline{F}]$-connected set $C = \{x_k \in \mathcal{X} | 0 < i \leq k \leq j \leq N\}$ is

$$\underline{P}(C) = \max\{0, \underline{F}(x_j) - \overline{F}(x_{i-1})\}.$$

With $\overline{F}(x_0) = \underline{F}(x_0) = 0$. **Lower probability additive on disjoint union $E$ of $[\underline{F}, \overline{F}]$-connected set $C_k$** : $E = C_1 \cup \ldots \cup C_M$

$$\underline{P}_{[\underline{F},\overline{F}]}(E) = \sum_{k=1}^{M} \underline{P}_{[\underline{F},\overline{F}]}(C_k).$$

Using boolean sub-algebra $\mathcal{H}$ induced by focal sets, we have, for any $A$, $\underline{P}_{[\underline{F},\overline{F}]}(A) = \underline{P}_{[\underline{F},\overline{F}]}(A_*)$ with $A_*$ its maximal inner approx. in $\mathcal{H}$. If $A_* = C_1 \cup \ldots \cup C_M$ and $C_i = \{x_{\underline{i}}, \dots, x_{\overline{i}}\}$, that

$$\underline{P}(A) = \sum_{i=1}^{M} \max\{0, \underline{F}(x_{\overline{i}}) - \overline{F}(x_{\underline{i}-1})\}.$$

Upper probabilities are obtained via duality $\overline{P}(A) = 1 - \underline{P}(A^c)$

**example:**

$$x_1 \leq_{[\underline{F},\overline{F}]} x_2 \leq_{[\underline{F},\overline{F}]} x_3 \simeq_{[\underline{F},\overline{F}]} x_4 \leq_{[\underline{F},\overline{F}]} x_5 \leq_{[\underline{F},\overline{F}]} x_6 \simeq_{[\underline{F},\overline{F}]} x_7$$

$$A = \{x_1, x_3, x_4, x_5, x_6\}$$

$$A_* = \{x_1\} \cup \{x_3, x_4, x_5\}$$

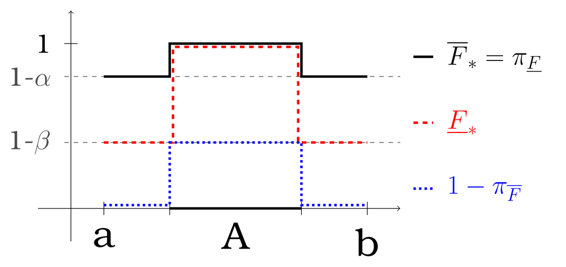
$$\underline{P}(A) = \underline{P}(A_*) = \max(0, \underline{F}(x_1) - \overline{F}(x_0)) + \max(0, \underline{F}(x_5) - \overline{F}(x_2))$$

## 2. Elicitation and representation

For $i = 1, \dots, N$, let denote $\alpha_i$, $\beta_i$ the bounds $\overline{F}(x_i)$, $\underline{F}(x_i)$ and $A_i$ the sets $\{x_1, \dots, x_i\}$. A gen. p-box can then be **elicited as upper/lower confidence levels on nested sets/intervals**.

→ Extend experts giving (imprecise) percentiles (p-boxes)
→ Extend experts giving only lower bound (possibility dist.)

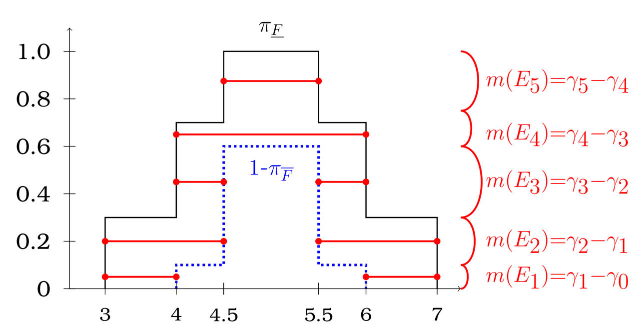
**Example 1:** Given a variable $\Theta$ with domain $[a, b]$ and the question "is $\Theta$ in $A$?", expert gives two bounds $\alpha \leq P(A) \leq \beta$




And $m(A) = \alpha$; $m([a, b]) = \beta - \alpha$; $m([a, b] \setminus A) = 1 - \beta$.

**Example 2:** An expert provide his opinion on a pH value in the form

$$0.3 \leq P(\text{pH} \in [4.5, 5.5]) \leq 0.6; \quad 0.7 \leq P(\text{pH} \in [4, 6]) \leq 0.9; \quad 1 \leq P(\text{pH} \in [3, 7]) \leq 1.$$




## 3. Conditioning

Given event $B = \{x_{b_1}, \dots, x_{b_M}\}$, two possible conditioning : Dempster's and Walley's

- **Dempster:** cond. upper measure, for $A \subseteq \mathcal{X}$

$$\overline{P}_{[B]}(A) = \frac{\overline{P}(A \cap B)}{\overline{P}(B)},$$

lower measure obtained by duality

**Proposition 1** *$\underline{P}_{[\underline{F},\overline{F}]}$ induced by a gen. p-box. Lower measure $\underline{P}_{[B]}$ **obtained by Dempster's conditioning stems from a gen. p-box** $[\underline{F}, \overline{F}]_{[B]}$ defined on $\mathcal{X} \cap B$ and yielding the restriction of $\leq_{[\underline{F},\overline{F}]}$ to elements $x \in B$.*

⇒ If $B_i = \{x_{b_1}, \dots, x_{b_i}\}$, sufficient to compute $\overline{P}_{[B]}(B_i)$, $\underline{P}_{[B]}(B_i)$ for $i = 1, \dots, M$. In particular, if $B$ $[\underline{F}, \overline{F}]$-connected, then, for $i = 1, \dots, M$

$$\overline{P}_{[B]}(B_i) = \frac{\overline{F}(x_{b_i}) - \underline{F}(x_{b_1 - 1})}{\overline{F}(x_{b_M}) - \underline{F}(x_{b_1 - 1})} = \overline{F}_{[B]}(x_{b_i}),$$

$$\underline{P}_{[B]}(B_i) = \frac{\underline{F}(x_{b_i}) - \underline{F}(x_{b_1 - 1})}{\overline{F}(x_{b_M}) - \underline{F}(x_{b_1 - 1})} = \underline{F}_{[B]}(x_{b_i}).$$

- **Walley:** cond. lower measure, for $A \subseteq \mathcal{X}$

$$\underline{P}_{|B}(A) = \frac{\underline{P}(A \cap B)}{\underline{P}(A \cap B) + \overline{P}(A^c \cap B)}.$$

$\underline{P}_{|B}$ **cannot, in general, be modeled by a gen. p-box**

## 4. Propagation

Propagating through function $f$ by computing image of sets, **3 methods giving $\neq$ random sets**:

- propagating $\alpha_i \leq P(f(A_i)) \leq \beta_i$ into $\alpha_i \leq P(f(A_i)) \leq \beta_i$

$$\left.\begin{array}{l} \alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j \end{array}\right\} \begin{array}{l} m(f(A_{i+1}) \setminus f(A_j)) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j). \end{array} \Rightarrow \mathcal{P}_{f([\underline{F},\overline{F}])}$$

with $\theta \in [0, 1]$. Gives **Inner approximation, low complexity.**

- Propagating sets $E_i$ of equivalent random set

$$\left.\begin{array}{l} \alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j \end{array}\right\} \begin{array}{l} m(f(A_{i+1} \setminus A_j)) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j); \end{array} \Rightarrow \mathcal{P}_{f((m,\mathcal{F}))}$$

with $\theta \in [0, 1]$. Gives **exact result, high complexity.**

- Propagating separately $\pi_{\overline{F}}, \pi_{\underline{F}}$ separately t

$$\left.\begin{array}{l} \alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j \end{array}\right\} \begin{array}{l} m(f(A_{i+1}) \setminus f(A_j^c)^c) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j). \end{array} \Rightarrow \mathcal{P}_{f(\pi_{\underline{F}}, \pi_{\overline{F}})}$$

with $\theta \in [0, 1]$. Gives **outer approximation, Intermediate complexity.**

$$\mathcal{P}_{f([\underline{F},\overline{F}])} \subseteq \mathcal{P}_{f((m,\mathcal{F}))} \subseteq \mathcal{P}_{f(\pi_{\underline{F}}, \pi_{\overline{F}})},$$

The inclusions turning into equalities when $f$ is injective (limiting in practical cases)

## 5. merging rules

$S$ different gen. p-boxes $[\underline{F}, \overline{F}]_1, \dots, [\underline{F}, \overline{F}]_S$, said to form a comonotonic set if $\underline{F}_i, \overline{F}_i$, $i = 1, \dots, S$, are all comonotonic.

### idempotent merging rules

- **Conjunction**: define $[\underline{F}, \overline{F}]_\cap$, for any $x \in \mathcal{X}$ , as

$$\underline{F}_\cap(x) = \max_{i=1,S} \underline{F}_i(x) \text{ and } \overline{F}_\cap(x) = \min_{i=1,S} \overline{F}_i(x).$$

- **Disjunction**: define $[\underline{F}, \overline{F}]_\cup$, for any $x \in \mathcal{X}$ , as

$$\underline{F}_\cup(x) = \min_{i=1,S} \underline{F}_i(x) \text{ and } \overline{F}_\cup(x) = \max_{i=1,S} \overline{F}_i(x).$$

- **Mean**: consider $\lambda_1, \dots, \lambda_S$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{S} \lambda_i = 1$.define $[\underline{F}, \overline{F}]_\Sigma$, for any $x \in \mathcal{X}$ , as

$$\underline{F}_\Sigma(x) = \sum_{i=1}^{S} \lambda_i \underline{F}_i(x) \text{ and } \overline{F}_\Sigma(x) = \sum_{i=1}^{S} \lambda_i \overline{F}_i(x)$$

⇒ retrieve classical p-box merging and possibilistic merging as special cases

⇒ in general, merging result is not a gen. p-box but a cloud (lost of monotonicity).

**Proposition 2** *Let $\mathcal{P}_{[\underline{F},\overline{F}]_1}, \dots, \mathcal{P}_{[\underline{F},\overline{F}]_S}$ be the sets of probabilities induced by $[\underline{F}, \overline{F}]_1, \dots, [\underline{F}, \overline{F}]_S$. Then, the following inclusions hold*

$$\mathcal{P}_{[\underline{F},\overline{F}]_\cap} \subseteq \bigcap_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i}; \quad \mathcal{P}_{[\underline{F},\overline{F}]_\cup} \supseteq \bigcup_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i},$$

*first inclusion turning into an equality when generalised p-boxes form a comonotonic set.*

### non-idempotent merging rules

Extending idempotent rule by **using a t-norm $\top$ and its dual triangular conorm $\perp$**, possibly restricted to associative copulas. Disjunction and conjunction then respectively become

$$\underline{F}_\top(x) = \perp_{i=1,S} \underline{F}_i(x); \; \overline{F}_\top(x) = \top_{i=1,S} \overline{F}_i(x).$$

$$\underline{F}_\perp(x) = \top_{i=1,S} \underline{F}_i(x); \; \overline{F}_\perp(x) = \perp_{i=1,S} \overline{F}_i(x).$$

⇒ Allow taking source (in)dependencies into account ?
⇒ Inclusions of Proposition 2 remaining valids.
⇒ Equivalent to separately apply $\top$ (conjonction) or $\perp$ (disjunction) to $\pi_{\overline{F}}, \pi_{\underline{F}}$
⇒ Contrary to possibilistic case, no clear relation between $\top$=product and Dempster's rule of combination

## Conclusions

- Generalized p-boxes **not very stable** under information processing (propagation, merging, . . . ), as result often not a generalized p-boxes, except under specific assumptions.
- **Main interests** lies in **uncertainty elicitation/representation**, thanks to their interpretation in terms of lower/upper confidence bounds, and in using them as **approximation alleviating computational burden**.
- Other potential interests: optimization under uncertainty using convex confidence regions [2], kernel signal-filtering with imprecise probabilities.

## References


[1] S. Destercke, D. Dubois and E. Chojancki. Unifying practical uncertainty representations - I: Generalized p-boxes. *Int. J. Approx. Reasoning*, 49:649–663, 2008.

[2] M. Fuchs and A. Neumaier. Potential based clouds in robust design optimization. *J. of Statistical Theory and Practice*, 3:225–238, 2009.

[3] A. Neumaier. Clouds, fuzzy sets and probability intervals. *Reliable Computing*, 10:249–272, 2004.